Polynomial curve registration for matching point clouds of different scales

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Abstract—Two point clouds of the same scene reconstructed by different techniques may differ in scale. A new scale registration algorithm is proposed in the paper to find the scale difference between two point clouds. First, principle components analysis is applied on descriptors of the point cloud to derive different cumulative contribution rates (CCR) in different neighbor regions. The discrete CCR of each point cloud are then fitted by polynomial scale curves (PSC). The unknown scale is formulated as a solution minimizing the scale difference between two PSCs. Finally, the optimal solution is explicitly formulated as one eigenvalue of a companion matrix. Experimental results prove that the proposed algorithm achieve higher accuracy than existing methods.

Keywords- point cloud; scale; polynomial curve; eigenvalue

I. INTRODUCTION

For different point clouds of the same scene captured by different sensors, their scales may differ due to deployment of different 3D reconstruction techniques [1]. For example, without baseline information and reference object, structure-from-motion with a single camera cannot provide the desired scale of the scene [2]. As such, before fusing point clouds from different sensors, identifying the relative scale between two point clouds becomes one important task [3-5].

Given the scale between two point clouds, they can be aligned together based on scale invariant descriptors such as Spin Image [6], Fast Point Feature Histograms (FPFH) [7], Rotational Projection Statistics (RoPs) [8], and Signature of Histograms of Orientations (SHOT) [9]. Such descriptors, based on a radius defining a neighbor for each point, does not encode the scale information of the point cloud, thus cannot be directly used for aligning two point clouds of different scales.

The unknown scale between the two point clouds can be roughly estimated as the difference between their resolutions [10]. Since the resolution was estimated as the median of distances between points, the accuracy will decrease when the point density varies. To overcome such limitations, keyscale method [3] was proposed applying principle components analysis (PCA) on spin images [6]. The PCA provided the cumulative contributions rates (CCR) for each point cloud. A keyscale for the point cloud is a minimum of CCR encoding its scale. Then, the unknown scale is estimated as the difference between the two keyscales. A dense set of radii was required in order to capture the correct keyscale in each CCR. Scale ratio method [4] further improved the result by matching the whole CCRs instead of their minimums. However, since the algorithm [4] forced a point to point matching between the two CCRs, the accuracy also relies on a dense set of radii, which is computationally expensive. The accuracy may drop with a sparse set of radii.

In this paper, we proposed a method to derive the unknown scale by matching polynomial curves instead of discrete CCRs, avoiding the point-to-point matching error in [4]. Then, by reformulating the best matching problem into finding the best polynomial roots, the proposed method then turns the scale registration in to searching for the best eigenvalue of a companion matrix. Finally, experiments and comparisons are presented to illustrate its performance.

II. SCALE REGISTRATION

A. Descriptor Analysis

The proposed scale registration method is based on different 3D feature descriptors, among which Spin Image [6], FPFH [7], RoPs [8], and SHOT [9] are tested in this paper. For a point \( p \) in a point cloud \( P \), its descriptor \( p \), formulates its neighbor points \( \{ q \in P \| \| q - p \| \leq r \} \) into a histogram of size \( t \). We extend the calculation of CCR [3, 4] to different descriptors in TABLE I.

<table>
<thead>
<tr>
<th>TABLE I. DESCRIPTORS TESTED IN THIS PAPER</th>
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<tbody>
<tr>
<td>Size ( t )</td>
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</table>

The descriptor \( p_r \) varies as the radius \( r \) changes

\[
 r \in \{ r_1, r_2, \ldots, r_m \},
\]

where \( m \) is the number of radii for calculating descriptors. Put all descriptors of the same radius together into

\[
P_r = \{ p \in P \},
\]

and perform PCA on \( P_r \) to derive \( t \) eigenvalues

\[
\lambda_{i,1} \geq \lambda_{i,2} \geq \ldots \geq \lambda_{i,t} \geq 0.
\]

The CCR for the point cloud \( P \) are

\[
\eta_{ir} = \frac{\sum_{j=1}^{t} \lambda_{ij}}{\sum_{j=1}^{t} \lambda_{ij}}, \quad r \in \{ r_1, r_2, \ldots, r_m \}, \quad i \in I = \{ 1, 2, \ldots, t \}.
\]

Figure 1 plots different CCRs using different descriptors for the Stanford bunny [11], where each curve connects the discrete points

\[
\eta_i = \left( \left( r_j, \eta_{ir} \right) \right)_{j=1}^{m}.
\]
where \( f_i \) is a polynomial curve fitting \( \eta_i \) in Eq.(5). Denote \( \{ f_i(x) \}_{i=1}^n \) and \( \{ g_i(x) \}_{i=1}^n \) as PSCs for \( P \) and \( P' \), respectively. Then, the unknown scale \( s \) is the one that can provide a best match for \( \{ f_i(xs) \}_{i=1}^n \) and \( \{ g_i(x) \}_{i=1}^n \), which will be presented in the following section.

C. Polynomial Function Matching

For \( i \in I \), denote PSCs for two different point clouds as

\[
\begin{align*}
  f_i(x) &= \sum_{j=0}^{n} a_{ij} x^j, \ x \in [0, F_i], \\
  g_i(x) &= \sum_{j=0}^{n} b_{ij} x^j, \ x \in [0, G_i].
\end{align*}
\]

where

\[
\begin{align*}
  F_i &= \max(\arg\min \eta_{ir} = 1, \arg\max \eta_{ir} < 1), \\
  G_i &= \max(\arg\min \eta_{ir}' = 1, \arg\max \eta_{ir}' < 1), \\
  M_i &= \max(F_i, G_i).
\end{align*}
\]

Define

\[
  f_i(xx) = \sum_{j=0}^{n} a_{ij} s^j x'.
\]

The distance between \( f_i(xx) \) and \( g_i(x) \) is given by

\[
D(s) = \sum_{i=1}^{n} \int_{0}^{M_i} (f_i(xx) - g_i(x))^2 \, dx.
\]

Then the scale between the two point clouds can be estimated as

\[
s = \arg\min_{s} D(s).
\]

The following theorem proves that the solution of Eq.(11) can be derived by searching at most \( (2n-1) \) eigenvalues.

**Theorem 1:** The solution for Eq.(11) is

\[
s = \arg\min_{s \in \{s \in I \}} D(s),
\]

where \( s_i \) are real eigenvalues of the matrix

\[
\begin{pmatrix}
  d_{2n-3} & d_{2n-4} & \ldots & d_{2} & d_{1} \\
  d_{2n-4} & d_{2n-5} & \ldots & d_{3} & d_{2} \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  d_{3} & d_{4} & \ldots & d_{2} & d_{1} \\
  0 & 1 & \ldots & 0 & 0
\end{pmatrix}
\]

with

\[
d_i = \sum_{j=0}^{n} \sum_{k=0}^{n} \epsilon_{i,j,k} M^{j+k+2}.
\]

\[
\epsilon_{i,j,k} = \left\{ \begin{array}{ll}
  \sum_{l=0}^{\min(n-k,0)} (k+1)a_{ij+l}a_{l,n}, & 0 \leq l < n, \\
  0, & 1 \leq j \leq n, \quad 0 \leq i \leq n, \quad n \leq l \leq 2n-1.
\end{array} \right.
\]

**Proof:** Denote

\[
H_i(x, s) = f_i(xx) - g_i(x) = \sum_{j=0}^{n} a_{ij} s^j x' - \sum_{j=0}^{n} b_{ij} x^j,
\]

For the point cloud \( P \), the polynomial scale curves (PSC) are polynomial curves of degree \( n \)

\[
f_i(x) = \sum_{j=0}^{n} a_{ij} x^j, \ i \in I,
\]

As will present in Section III.B, the accuracy of such method also relies on a dense set of data in CCR, but it is invalid for descriptors without minimum in CCR (top right in Figure 1). Scale ratio [4] reconstructed the unknown scale \( s \) by forcing a match between \( \{r_i, \eta_{ir} \} \) and \( \{r'_i, \eta'_{ir} \} \) in the minimization problem

\[
\arg\min \sum_{i} \sum \left( \frac{\eta_{ir}}{s \cdot r_i} - \frac{\eta'_{ir}}{r'_i} \right)^2.
\]

For the point cloud \( P \), the polynomial scale curves (PSC) are polynomial curves of degree \( n \)

\[
f_i(x) = \sum_{j=0}^{n} a_{ij} x^j, \ i \in I,
\]
with partial differential equation as
\[
\frac{\partial H_i(x,s)}{\partial s} = \sum_{j=0}^{k} j a_j s^{j-1} x^j = \sum_{j=1}^{k} j a_j s^{j-1} x^j + \sum_{j=k+1}^{k} (k+1) a_j s^j x^{j-1}.
\]

Coupling \( F_i(x,s) = H_i(x,s) \) with Eq.(15) and Eq.(16) gives
\[
\frac{1}{2} \frac{\partial F_i(x,s)}{\partial s} = H_i(x,s) \frac{\partial H_i(x,s)}{\partial s} = \sum_{j=0}^{k} a_j x^j \left( \sum_{j=0}^{k} b_j x^j \right) (k+1) a_j x^j + \sum_{j=0}^{k} b_j x^j (k+1) a_j x^{j-1} s^j x^{j-1}.
\]

where
\[
h_j(x) = \max_{l=1,2 \ldots n} (k+1)a_j x^j - \left( \sum_{j=0}^{k} b_j x^j \right) (k+1)a_j x^{j-1} (0 \leq j < n, 0 \leq l \leq n-1, n \leq l \leq 2n-1).
\]

The solution for Eq.(11) satisfies \( \frac{\partial D(s)}{\partial s} = 0 \), i.e.
\[
\sum_{j=0}^{2n-1} d_j s^j = 0 \quad (17)
\]

The roots of the polynomial Eq.(17) are real eigenvalues of its companion matrix Eq.(13) [12]. Thus, the optimal solution for Eq.(11) is the eigenvalue in Eq.(12)

D. Matching Algorithm

The proposed scale-matching algorithm is summarized as follow:

Input: two different point clouds \( P_1 \) and \( P_2 \)
Output: the scale \( s \)
Step 1: Initialization
Choose a descriptor and initialize the descriptor size \( t \)
Specify \( n, m \) and set \( I = \{1, 2, \ldots , t\} \), \( J = \{1, 2, \ldots , n\} \)
Step 2: PSC reconstruction
For \( P \in \{ P_1, P_2 \} \)
Estimate the resolution \( \mu \) of \( P \)
Initialize \( \{r_i, r_2, \ldots , r_m\} \) according to \( \mu \)
For \( r \in \{r_1, r_2, \ldots , r_m\} \)
Perform PCA on \( P_r \) to get \( \{ \lambda_i \}_{i=1}^t \) in Eq.(3)
Calculate CCR in \( \{ \eta_i \}_{i=1}^t \) Eq.(4)
For \( i \in I \)
Form discrete points \( \eta_i \) in Eq.(5)
Polynomial fit \( \eta_i \) to get PSC \( f_j(x) \) in Eq.(7)
Step 3: Scale reconstruction
Initialize \( \{a_i, b_i\}_{i=1, j=1} \) and \( \{M_i\}_{i=1} \) in Eqs.(8) and (9)
Derive \( \{d_i\}_{i=1}^{2n-1} \) in Eq.(14)
Form the companion matrix Eq.(13)
Calculate real eigenvalues \( \{s_i\} \) of Eq.(13)
Obtain the scale \( s \) according to Eq.(12)

III. EXPERIMENTS

The proposed algorithm is applied on different 3D point clouds to demonstrate its higher accuracy over existing methods. The same data used in [4] are adopted to show the efficiency of the proposed method over existing methods [4, 10]. The point cloud \( P_1 \) is a Stanford bunny [11] with 69451 points, and the point cloud \( P_2 \) is a scaling of \( P_1 \) by the factor of 5. The accuracy of the estimated scale is
\[
\delta = \delta(m,n) = \left| \frac{s - \delta}{5} \right|
\]

The accuracy \( \delta \) may vary for different \( m \) and \( n \). Thus, experiments are conducted to select suitable \( n \) and \( m \). Moreover, noises are applied onto \( P_1 \) and \( P_2 \) to demonstrate its performance over different level of noises. Finally, the scale registration accuracies based on different descriptors are compared and analyzed.

A. Selection of polynomial degree \( n \)

This experiment based on Stanford bunny and Spin Images is to study how the PSC degree \( n \) affects the accuracy \( \delta \). For each \( m \in \{11, 12, \ldots , 17\} \), different scales are obtained for different degrees \( n \in \{2, 3, \ldots, 7\} \), forming an accuracy curve in Figure 2. On the other hand, for each \( n \), the variation in the accuracy using different \( m \) is defined as
$$\delta' = \delta'(n) = \max_{m} \delta(m, n) - \min_{m} \delta(m, n).$$

Figure 3 plots the results of $\delta'(n)$. According to the Figure 2 and Figure 3, $n=3$ is selected for a high accuracy ($\delta \geq 98.8\%$) with a low variation ($\delta' \approx 0.26\%$).

Figure 2  
Accuracy $\delta$ with different $n$: different curves for different $m$.

Figure 3  
Accuracy variation $\delta'$ using different $n$.

B. Performance over different $m$

This section will show how the proposed method outperform the scale ratio [4] based on the same sets of radii for Spin Images. The accuracies $\delta$ for different methods using different $m$ are presented in different curves in Figure 4. The red curve from our result is better than the blue curve from scale ratio [4]. Based on the same CCRs, the proposed method can achieve higher accuracy.

Figure 4  
Accuracy comparison of scale ratio [4] and our method: blue curve with circles for the result of [4]; red curve with squares for the our results.

Figure 5 and Figure 6 compare the matching results of both methods based on a given $m$. Scale ratio [4] tried to match the discrete points, i.e. matching the red circles to the blue circles (the left bottom image in Figure 5). The two curves may not match very well. On the other hand, our method tries to match the two curves leaving the discrete points not well matched (the right bottom image in Figure 5). The result indicates that the curve based matching provides a higher accuracy (Figure 4). Especially, when $m$ is smaller, the point mis-matching error in [4] may highly reduce the accuracy (Figure 6).

For a given $m$, the descriptor on each point need to be calculated $m$ times for $m$ different radii. Thus, smaller $m$ leads to faster computation. The computation time for $m=10$ is only $56\%$ of that for $m=18$. Meanwhile, according to Figure 4, $m=10$ can achieve similar accuracy as $m=18$. In the following, $m=10$ is assumed.

Figure 5  
Point based matching VS curve based matching at $m=18$: circles for discrete points $\eta$; blue data for $G_i(s)$; red data for $F_i(s\eta)$; left column for scale ratio [4] (98.18% accuracy); right column for our results (99.34% accuracy); first row for overall view; second row for zoom view.

Figure 6  
Point based matching VS curve based matching at $m=10$: circles for discrete points $\eta$; blue data for $G_i(s)$; red data for $F_i(s\eta)$; left image for scale ratio [4] (81.93% accuracy); right image for our results (97.03% accuracy).

C. Performance over noises

This experiment adds noise to the point cloud to demonstrate the performance of the proposed method in different noise levels. For each noise level $L$ and point cloud resolution $\mu$, Gaussian noise with mean zero and standard deviation $\mu \cdot L$ was added to each point. Figure 7 shows that Gaussian noise does not significantly affect the accuracy of the proposed algorithm, which outperform [4].
D. **Performance of descriptors**

The first step of the algorithm is to select a descriptor from TABLE I. Different descriptors may lead to different accuracies. With \( m = 10 \), the accuracies applying different descriptors are listed in TABLE II. Our method can provide a higher accuracy.

<table>
<thead>
<tr>
<th>Descriptors</th>
<th>Scale ratio [4]</th>
<th>Our method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spine Image [6]</td>
<td>81.93%</td>
<td>97.03%</td>
</tr>
<tr>
<td>FPFH [7]</td>
<td>80.14%</td>
<td>91.40%</td>
</tr>
<tr>
<td>RoPS [8]</td>
<td>80.19%</td>
<td>90.50%</td>
</tr>
<tr>
<td>SHOT [9]</td>
<td>79.25%</td>
<td>93.90%</td>
</tr>
</tbody>
</table>

IV. **CONCLUSIONS**

In this paper, a new scale registration method is proposed for aligning point clouds of different scales. For each point cloud, PCA is performed over descriptors under different radii to obtain the discrete CCR. The polynomial fitting on CCR gives continuous PSC, which is used for scale registration. The unknown scale is formulated as the one giving the best matching between two PSCs instead of two CCRs. It is proved that the best scale is one eigenvalue of the companion matrix. Experiments results indicate that the proposed method can provide higher accuracy than the existing method.

REFERENCES


