A Study of Plane Waves Propagating in a Pipe Filled with Real Gas

Chen Runchang[1], Khoo Wee Yang David[2], Cui Shan[2], and Khoo Boo Cheong[1]

[1]Mechanical Engineering Department, National University of Singapore
[2]National Metrology Centre, Agency for Science, Technology and Research, Singapore

Abstract**—**At high pressure and low temperature, gases in pipelines behave as real gases rather than ideal gases. The Van der Waals equation was used to represent the properties of real gases and the Fourier-Bessel-based acoustic model was extended to real gas cases. The effects of frequency, Mach number and the radius of a pipe on the plane waves in real gases were investigated. The results indicated that plane waves travel more slowly in the real gas model than in the ideal gas model while the attenuation coefficient is higher in the real gas model. Higher frequency waves and a larger radius of pipe can strengthen the influence of the real gas. Flow with higher Mach number weakens the influence of the real gas in the downstream propagation while it intensifies the influence in the upstream propagation.

Keywords**—**Attenuation coefficient, Phase velocity, Real gas, Van der Waals equation.

I. INTRODUCTION

Pipe acoustics have been widely used in fields of pipe flow measurement and leakage detection. The speed of acoustic wave and attenuation coefficient are both very important parameters to characterize waves traveling in a pipe. Under accurate predictions of the speed of acoustic wave and attenuation coefficients, the velocity of flow and location of pipeline leakages can be quantified with satisfactory accuracy.

To simulate acoustic wave propagation in pipes filled with gases, the conservation laws of mass, momentum, and energy are often applied as governing equations. The ideal gas law is also generally applied to represent the relationship between pressure, density, and temperature and finally used to close the system of equations. Kirchhoff [1] was perhaps the first to fully describe pipe acoustics and give a complete but not so easily applicable solution. To solve the so-called Kirchhoff's model, some hypotheses are often utilized. Neglecting the moving flow, Rayleigh [2] proposed a model which has a more convenient solution. Assuming that the radius of a pipe is small enough and considering only the high-frequency waves, Dokumaci [3, 4] studied the influence of uniform and shear moving flow.

To overcome the limitation of narrow pipe and consider moving flow as well as thermal effects simultaneously, Chen et al. [5] proposed a Fourier-Bessel-based model to study the problem of axisymmetric wave propagation in ideal gas. Afterwards, several works were conducted to investigate the influence of shear flow and thermoviscous dissipation on axisymmetric waves traveling in pipe flow [6-8].

Although the influence of the moving flow and thermal effects have been taken into consideration, it should be noted that the gases applied are still assumed to be ideal or perfect gases when studying the sound wave propagations in pipes filled with gases. The law of ideal gas neglects volume occupied by gas molecules and the intermolecular forces. However, in practice, most gases do not necessarily obey the law of ideal gas, especially under low temperature and high pressure. For example, when studying the characteristics of natural gas through high-pressure nozzles, a dramatic variation between ideal gas assumption and the real gas assumption has been demonstrated [9, 10]. Chen [11] also shows that the speed of sound in free space calculated by the real gas model (van der Waals equation) is different from that by the ideal gas law.

Therefore, to overcome the limitation of ideal gases, this paper intends to extend the model proposed by Chen et al. in [5] to real gas cases. The differences on phase velocity and attenuation coefficient between the ideal gas model and the real gas model will be shown, respectively.

II. FORMULATION

A. Conservation laws of Mass, Momentum, and Energy

Sound wave traveling in a straight pipe filled with a gas is subject to conservation laws of mass, momentum, and energy, which are given by [12, 13]:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{1}$$

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v}\right] = -\nabla p + \eta \nabla^2 \vec{v} + \left(\varsigma + \frac{\eta}{3}\right)\nabla(\nabla\cdot\vec{v}) \tag{2}$$

$$\rho c_p\left[\frac{\partial T}{\partial t} + (\vec{v}\cdot\nabla)T\right] = \kappa_{th}\nabla^2 T + \beta T\left[\frac{\partial p}{\partial t} + (\vec{v}\cdot\nabla)p\right] + \frac{\eta}{2}\sum_{ij=1}^{3}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\nabla\cdot\vec{v}\right)^2 + \varsigma(\nabla\cdot\vec{v})^2 \tag{3}$$

where ρ is density, $\vec{v}$ is velocity vector, t is time, p is pressure, η is dynamic viscosity, ς is volume viscosity, c_p is heat capacity at constant pressure, T is temperature, κ_{th} is thermal conductivity, β is the volume thermal expansion, and δ_{ij} is Kronecker delta function.

A. Author biography
B. Author biography

C. Author biography

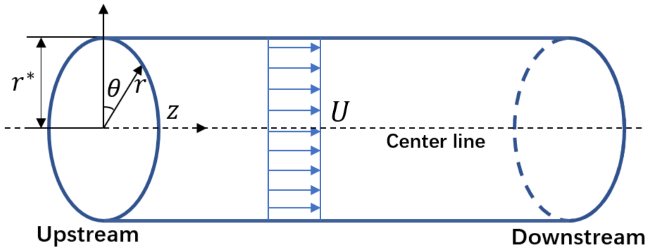

Fig. 1 Schematic diagram of the model

B. Assumptions

Pipe acoustics is a very complex problem, when all factors are considered. To simplify the problem and with particular attention on the effects of real gases in the model shown in Fig. 1, the following five assumptions are introduced.

1. The velocity profile of flow is uniform in steady state, $\vec{v} = [0,0,U]$. U is the velocity of flow in the z-direction.

2. The temperature of the wall is uniform and remains unchanged over time, $T_{wall} = constant$.

3. The wall of the pipe is rigid, which implies the wall only reflect sound waves and does not absorb any energy.

4. Only plane waves (axisymmetric waves) are considered in this paper.

5. Fluctuations of acoustic pressure, velocity, density, and temperature are very small.

The above five assumptions are exactly the same as those in Ref. [5].

C. State of Equation

After linearizing the conservation equations (1-3), the system of equations contains four equations with five unknown parameters $[\rho', p', v_r', v_z', T']$, where the prime $'$ indicates the acoustic variables and v_r', v_z' are fluctuations of the velocity in radial and axial directions, respectively. To close the system of equations, one more equation is required. The equation of state is a good choice. For gases under low pressure and high temperature, the law of ideal gas is reliable and can characterize the relationship between pressure and temperature

$$pV_m = RT \tag{4}$$

where V_m is the molar volume and R denotes the universal gas constant.

However, in most cases, gases in a high-pressure pipeline are not strictly ideal gases and thus the ideal gas law will lead to large deviation of gas properties from actual behaviors. Considering the volume of molecules and intermolecular interactions, Van der Waals proposed the Van der Waals equation to modify the ideal gas law [14] as follows:

$$p = \frac{RT}{(V_m - b)} - \frac{a}{V_m^2} \tag{5}$$

$$a = 27(RT_c)^2/64p_c \tag{6}$$

$$b = RT_c/8p_c \tag{7}$$

where T_c and p_c are critical temperature and critical pressure, respectively. The term a/V_m^2 and the parameter b are to modify the pressure and volume through considering attractive forces between molecules and volume of molecules, respectively. When $a = 0$ and $b = 0$, the Van der Waals equation will be reduced to the ideal gas law.

With the molar volume $V_m = M/\rho$ and the specific gas constant $R_s = R/M$, the equation of state for ideal gases and real gases can be re-arranged, respectively, as follows

Ideal gases: $p = \rho R_s T$ (8)

Van der Waals equation:

$$\left(p + \frac{\rho^2}{M^2}a\right)\left(1 - \frac{\rho}{M}b\right) = \rho R_s T \tag{9}$$

where M is molar mass.

D. Linearized Equation of State

To linearize the equation of state, perturbations p', T', ρ' are added to corresponding steady-state variables. After expanding Eqs. (8-9) and removing high-order parts, a linearized equation of state is written as the following form,

$$S_1 p' = S_2 T' + S_3 \rho'. \tag{10}$$

TABLE I

PARAMETERS FOR (10)

Type of gases	Parameters
Ideal gases:	$S_1 = 1$
	$S_2 = R_s\rho_0$
	$S_3 = R_s T_0$
Van der Waals:	$S_1 = \left(1 - \frac{\rho_0 b}{M}\right)$
	$S_2 = R_s\rho_0$
	$S_3 = \frac{R_s T_0}{(1-\rho_0 b/M)} - \frac{2a\rho_0}{M^2}\left(1 - \frac{\rho_0 b}{M}\right)$

where the subscript $'0'$ indicates the steady-state variables.

E. Calculation of Volume Thermal Expansion at Steady State β_0

For a perfect gas, based on the ideal gas law, the volume thermal expansion and the temperature have the relationship $\beta T = 1$ [13]. However, the relationship $\beta T = 1$ can result in a large deviation from properties of real gases. Therefore, the expression of volume thermal expansion needs to be derived.

The expression of volume thermal expansion β can be derived by differentiating the equation of state and keeping the pressure unchanged. Based on the Van der Waals equation, the volume thermal expansion β at steady state can be expressed as,

Van der Waals equation:

$$\beta_0 = \frac{1}{V_m}\frac{\frac{R}{V_m - b}}{\frac{RT_0}{(V_m - b)^2} - \frac{2a}{{V_m}^3}} \tag{11}$$

where $V_m = M/\rho_0$.

F. Governing Equations

With nondimensionalization $x = r/r^*$, where r^* is the radius of pipe, and based on the theory of the Fourier-Bessel function, the final governing equations can be obtained as follows. Boundary conditions are the same as those in [5] and more details can also be found in [5]. These are

$$(HR)_m C_m^r + \sum_{n=1}^{\infty}\left[\frac{i\lambda_n^z r^* k_0 K}{\rho_0}\left(\zeta+\frac{\eta}{3}\right)C_n^z + \frac{\lambda_n^z S_3 r^* k_0 K}{S_1\omega(1-KM)}C_n^z + \frac{\lambda_n^z S_2 C_n^T}{S_1\rho_0}\right](RZ)_m^n = 0, \tag{12}$$

$$(HZ)_m C_m^z - \frac{iS_2 r^{*2} k_0 K C_m^T}{S_1\rho_0} + \sum_{n=1}^{\infty}\left[\frac{\lambda_n^r S_3 r^* k_0 K}{S_1\omega(1-KM)} + \frac{i\lambda_n^r r^* k_0 K}{\rho_0}\left(\zeta+\frac{\eta}{3}\right)\right](ZR)_m^n C_n^r = 0, \tag{13}$$

$$(HT)_m C_m^T - \frac{iS_3\beta_0 T_0}{S_1 c_p} r^{*2} k_0 K C_m^z + \sum_{n=1}^{\infty}\frac{\lambda_n^r S_3 \beta_0 T_0 r^*}{S_1 c_p}(ZR)_m^n C_n^r = 0, \tag{14}$$

where

$$(HR)_m = i\omega r^{*2}(1-KM) + \frac{S_3}{iS_1\omega(1-KM)}(\lambda_m^r)^2 + \frac{1}{\rho_0}\left(\zeta+\frac{4\eta}{3}\right)(\lambda_m^r)^2 + \frac{\eta}{\rho_0}r^{*2}k_0^2K^2,$$

$$(HR)_m = i\omega r^{*2}(1-KM) - \frac{iS_3 r^{*2}k_0^2K^2}{S_1\omega(1-KM)} - \frac{\eta}{\rho_0}(\lambda_m^z)^2 + \frac{r^{*2}k_0^2K^2}{\rho_0}\left(\zeta+\frac{4\eta}{3}\right),$$

$$(HT)_m = i\omega r^{*2}\left(1-\frac{S_2\beta_0 T_0}{S_1\rho_0 c_p}\right)(1-KM) + \frac{\kappa_{th}}{c_p\rho_0}(\lambda_m^z)^2 + \frac{\kappa_{th}}{c_p\rho_0}r^{*2}k_0^2K^2,$$

$$(RZ)_m^n = -\frac{2}{J_2^2(\lambda_m^r)}\int_0^1 J_1(\lambda_n^z x)J_1(\lambda_m^r x)x\,dx,$$

$$(ZR)_m^n = \frac{2}{J_1^2(\lambda_m^z)}\int_0^1 J_0(\lambda_n^r x)J_0(\lambda_m^z x)x\,dx.$$

Here, $C_m^r, C_m^Z, C_m^T, C_n^r, C_n^Z, C_n^T$ are the assigned coefficients which can be calculated if the acoustic perturbations are given, λ_m^r, λ_n^r are roots of the first-order Bessel function, λ_m^z, λ_n^z are roots of the zero-order Bessel function, the wave number is $k_0 = \omega/c_0$, the Mach number is $Ma = U/c_0$, ω is angular frequency, and c_0 is the speed of sound in free space.

G.Axial Number K

Equations (12-14) can be used to determine the value of axial number K. They are expanded respectively with respect to m $\left(m \in (1, L)\right)$ to a system of linear equations which can be re-written as

$$\vec{G}(K)\vec{X} = 0 \tag{15}$$

where matrix $\vec{X} = [C_1^r, \cdots, C_L^r, C_1^z, \cdots, C_L^z, C_1^T, \cdots, C_L^T]^{tr}$(the superscript *tr* denotes transposition). To obtain a non-trivial solution of $\vec{X}$, the determinant of $\vec{G}(K)$ must be zero.

$$\left|\vec{G}(K)\right| = 0 \tag{16}$$

The axial number K can be calculated by using numerical methods to solve Eq. (16), such as secant method and Newton's method.

The axial number K consists of a real and imaginary part. It can be expressed as $K = K_R + iK_I$. With known axial number K, the phase velocity c_p^* and attenuation coefficient A can be obtained by following equations [5],

$$c_p^* = c_0/K_R\,, A = |8.686k_0K_I|. \tag{17}$$

The dimensionless phase velocity can be expressed as,

$$1/K_R = c_p^*/c_0. \tag{18}$$

III. Validation

The validation will be conducted through comparing with results from the ideal-gas-based model in [5]. The basic parameters are $\rho_0 = 0.277\,\mathrm{kg/m^3}$, $c_p = 1184\,\mathrm{J/(Kg\cdot K)}$, $\kappa_{th} = 0.0786\,\mathrm{W/(K\cdot m)}$, $R_s = 287\,\mathrm{J/(Kg\cdot K)}$, $\gamma = 1.4$, $\varsigma = 0$, $\eta = 4.79\times 10^{-5}\,\mathrm{kg/(s\cdot m)}$, $T_{wall} = 1273\mathrm{K}$ and $L = 50$. For $r^* = 50\mathrm{mm}$, the frequency ranges from 185 Hz to 20000 Hz while for $r^* = 1\mathrm{mm}$, it covers a range of 185 Hz to 10000 Hz.

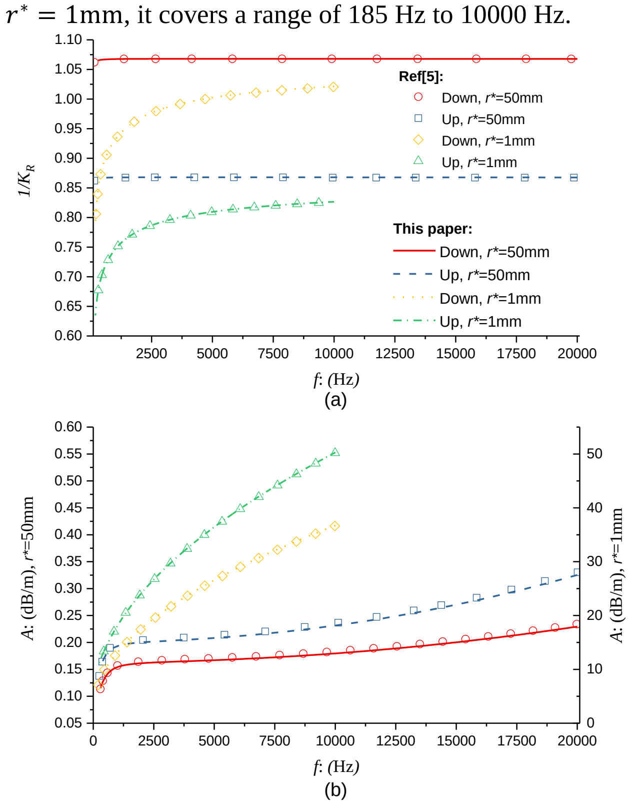

Fig. 2 Comparison of computed phase velocity and attenuation coefficient with Ref. [5]

As shown in Fig. 2, waves traveling in the same direction as pipe flow are marked by "down" while those in opposite direction as pipe flow are marked by "up". The phase velocity and attenuation coefficient from this paper can agree well with those from Ref. [5].

IV. Results and Analysis

This section concentrates on studying effects of frequency, flow velocity and the radius of a pipe on the phase velocity and attenuation coefficient when plane waves propagate in real gases. The medium studied is methane. The basic parameters are set as in Table II:

TABLE II
BASIC PARAMETERS

Symbol	Parameter	Value
ρ_0	density	$16.9\ \mathrm{kg/m^3}$
η	viscosity	$1.15 \times 10^{-5}\ \mathrm{kg/(s \cdot m)}$
T_{wall}	wall temperature	$298.15\ \mathrm{K}$
c_p	heat capacity at constant pressure	$2384.5\ \mathrm{J/(Kg \cdot K)}$
κ_{th}	thermal conductivity	$0.035\ \mathrm{W/(K \cdot m)}$
R	universal gas constant	$8.314\ \mathrm{J/(K \cdot mol)}$
γ	specific heat ratio	1.31
ς	bulk viscosity	$0\ \mathrm{kg/(s \cdot m)}$
c_0	speed of sound	$441.7\ \mathrm{m/s}$
p_c	critical pressure	$45.99\ \mathrm{bar}$
T_c	critical temperature	$190.56\ \mathrm{K}$
V_c	critical volume	$9.86 \times 10^{-5}\ \mathrm{m^3}$
M	molecular weight	$16.04 \times 10^{-3}\ \mathrm{kg/mol}$

To solve Eq. (16), the secant method is applied. After obtaining the axial number K, the phase velocity and attenuation coefficient can be calculated by Eqs. (17-18). The percentage change Δ is also defined to show the effect of real gas or the difference on phase velocity or attenuation coefficient between ideal gas model and real gas model. Here,

$$\Delta = \frac{(1/K_R)_{Van\ der\ Waals} - (1/K_R)_{ideal\ gas}}{(1/K_R)_{ideal\ gas}} \times 100\%$$

or

$$\Delta = \frac{(A)_{Van\ der\ Waals} - (A)_{ideal\ gas}}{(A)_{ideal\ gas}} \times 100\%.$$

The Δ could be positive or negative values, which can reveal the variation accordingly.

A. The effect of frequency

The influence of frequency on the phase velocity and attenuation coefficient for plane waves will be studied. The radius of pipe and the Mach number are set as $r^* = 0.05m$ and $Ma = 0.1$, respectively, which are the same as those found in [5].

Fig. 3 compares ideal gas and real gas models in terms of phase velocities of plane waves as frequency varies. As shown in Fig. 3, the phase velocities remain steady for both downstream and upstream propagations in an ideal gas and real gas. This implies that in a pipe with radius $r^* = 0.05m$, plane waves in the frequency range of 1-10000 Hz in the ideal gas and real gas are non-dispersive (velocities do not depend on frequency). It is also noticed that plane waves travel slower in real gas than in ideal gas. The reason is that under assumed pressure and temperature, the compressibility factor is lower than unity. This means that the real gas is more compressible as compared to ideal gas and distances between molecules in the real gas are larger than those in the ideal gas. It takes more time for molecules of real gas to collide with neighborhood ones. Therefore, plane waves propagate slower in the real gas.

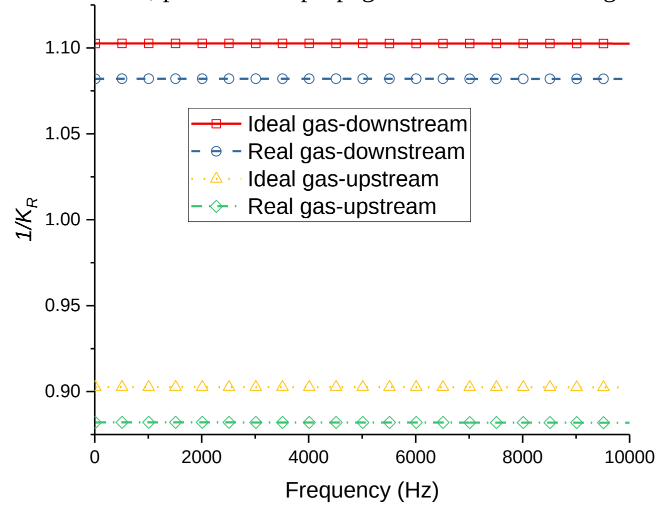

Fig. 3 Phase velocity of plane waves as a function of frequency

Fig. 4 shows the attenuation coefficient when plane waves ranging from 1 Hz to 10000 Hz travel in the ideal gas or real gas. It is clearly seen from Fig. 4 (a) that the attenuation coefficient in the real gas is higher than that in the ideal gas in both upstream and downstream propagations. Furthermore, as shown in Fig. 4 (b), with the increase of frequency, the effect of the real gas is more significant for waves traveling either downstream or upstream of pipes. The reason of having higher attenuation coefficient in the real gas is that the Van der Waals equation considers energy losses caused by collisions between gas molecules and the wall of the pipe. By contrast, the ideal gas law assumes the collisions are all perfectly elastic and thus no energy is consumed.

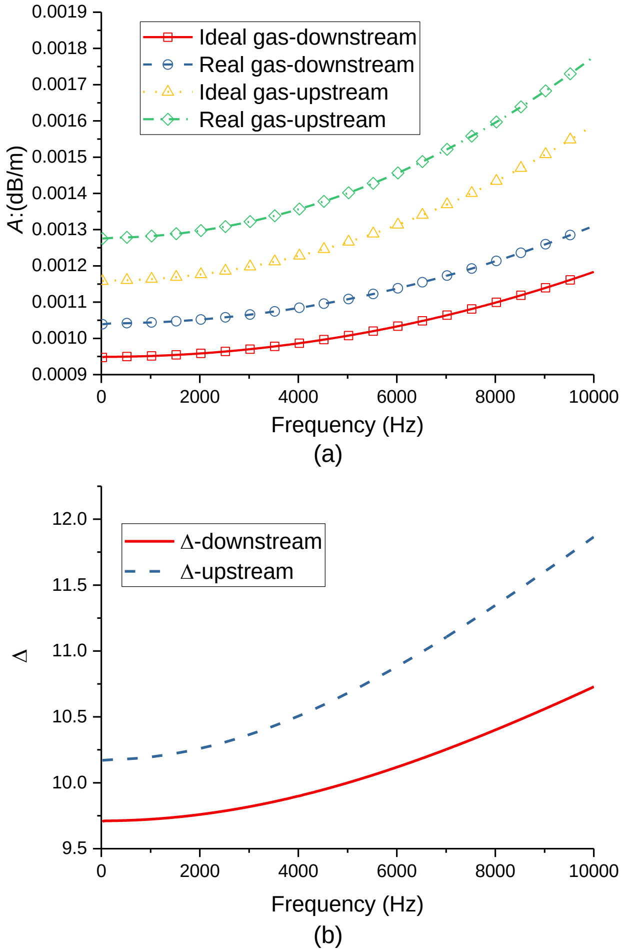

Fig. 4 Attenuation coefficient of plane waves as a function of frequency

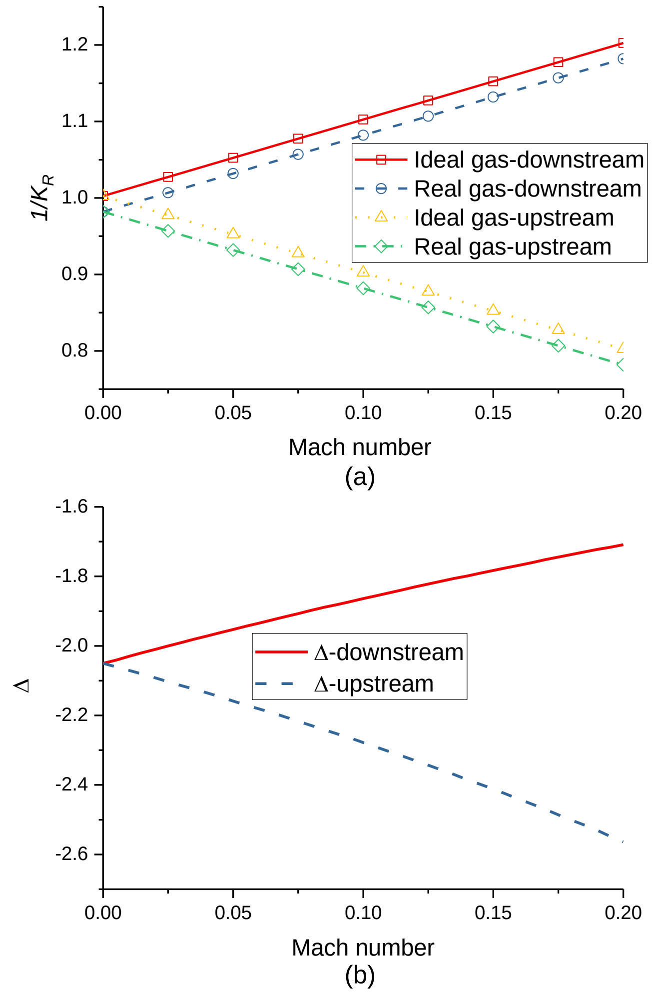

Fig. 5 Phase velocity of plane waves as a function of Mach number

B. The effect of the Mach number

This sub-section will analyze the influence of flow velocity on phase velocity and attenuation coefficient when plane waves propagate upstream and downstream of pipes. After nondimensionalization, the Mach number is used to represent uniform flow velocity. Radius of pipe and frequency of the wave are set as $r^* = 0.05$ m and $f = 10000$ Hz, respectively.

Fig. 5 depicts the relationship between the Mach number and phase velocity of plane waves. As presented in Fig. 5(a), the Mach number affects the phase velocity linearly in whichever type of the gas. High Mach number can increase the velocity in downstream propagation but decrease in upstream propagation. Plane waves still travel slower in the real gas than in the ideal gas despite the value of the Mach number. As observed from Fig. 5(b), with increasing Mach number the effect of real gas will be reduced for downstream flow while for upstream flow the effect becomes more significant.

It can be seen from Fig. 6 that as the Mach number increases, difference on attenuation coefficient between the two models shows similar trends as those on phase velocity. As demonstrated in Fig. 6(a), the increase of Mach number can reduce the attenuation coefficient in downstream propagation while it can increase the attenuation coefficient in the opposite propagation. As shown in Fig. 6(b), in downstream propagation, high Mach number can eliminate the effect of real gas but in upstream propagation it can strengthen the influence of the real gas.

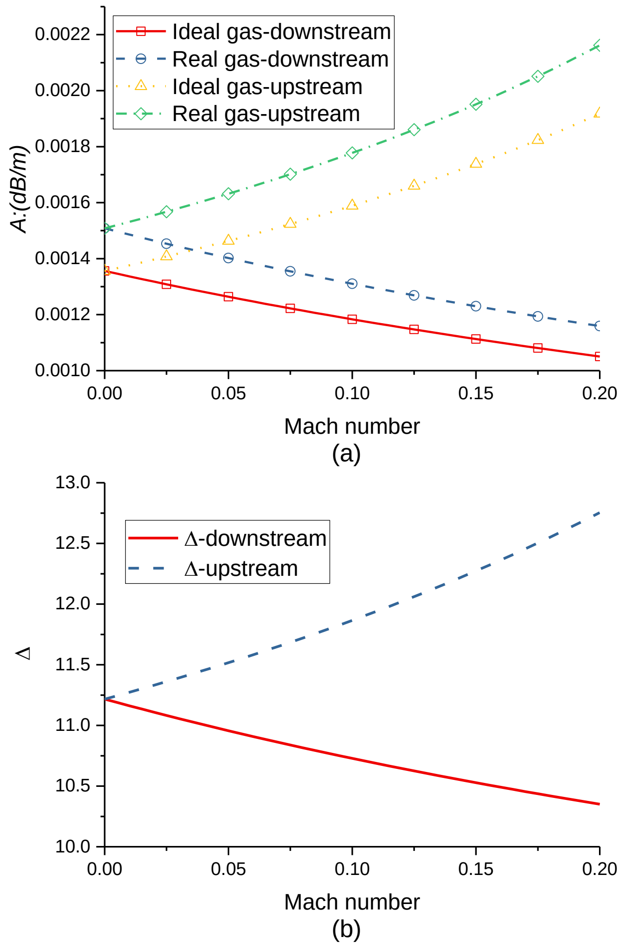

Fig. 6 Attenuation coefficient of plane waves as a function of Mach number

C. The effect of radius of the pipe

This section analyzes the effect of radius of the pipe on phase velocity and see the effect of real gas as the radius of the pipe changes. The Mach number of flow and frequency are $Ma = 0.1$ and $f = 10000$ Hz, respectively.

Fig. 7 shows that the radius of the pipe appears to have no significant influence on the phase velocity of plane waves despite propagation directions and types of gas. The phase velocity computed from the real gas model is still smaller than that from the ideal gas model. Fig. 7(b) shows that a larger radius of the pipe can strengthen the effect of real gas only very slightly.

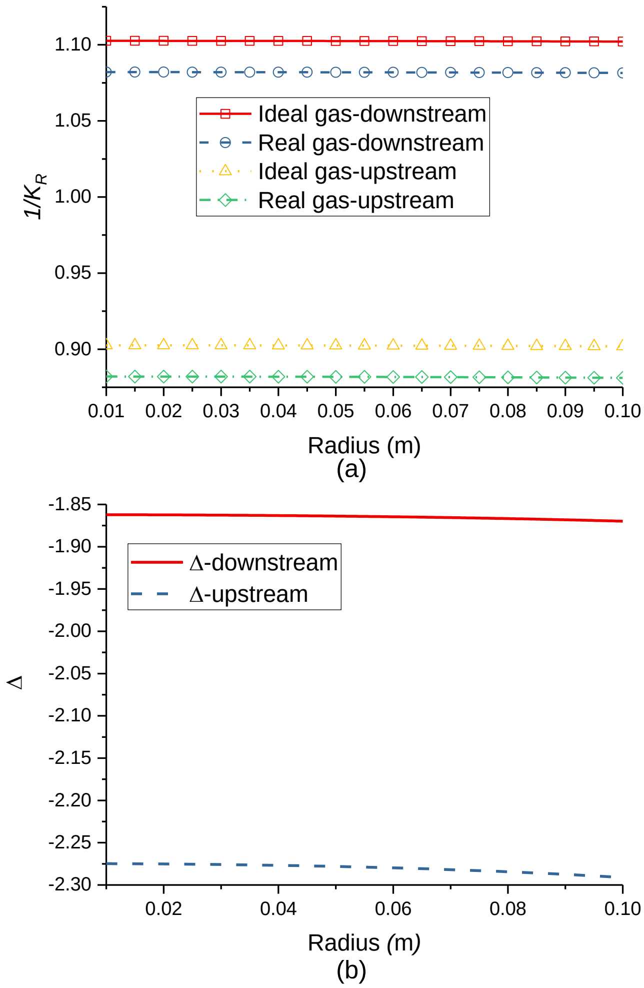

Fig. 7 Phase velocity of plane waves as a function of radius of the pipe

It is observed from Fig. 8 (a) that as the radius of the pipe becomes larger, in both downstream and upstream propagations, the attenuation coefficient decreases dramatically and then converges. Fig. 8 (b) shows that the difference between two models tends to be more dramatic with a larger radius of the pipe for both downstream and upstream cases.

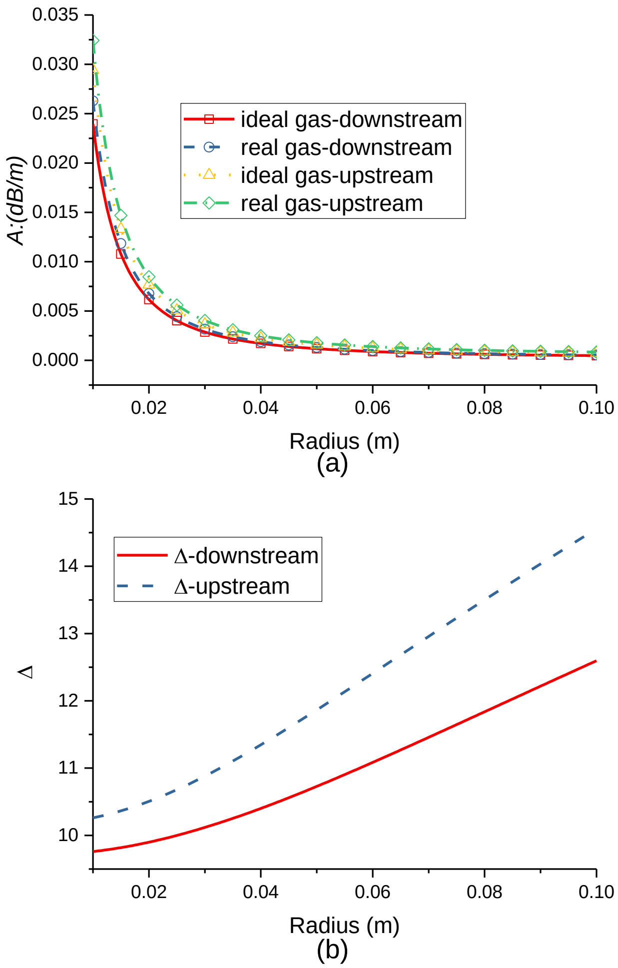

Fig. 8 Attenuation coefficient of plane waves as a function of radius of the pipe

V. CONCLUSION

This paper extends the ideal gas pipe acoustic model to real gas cases for natural gas pipeline flow applications. The equation of state, Van der Waals equation, is used to represent the characteristics of the real gas. Methane, the major component in natural gas, is studied as an ideal gas and a real gas, respectively. The effects of frequency, the Mach number and radius of pipe on plane wave propagation are investigated. Comparisons between two models are also presented:

I. Plane waves travel more slowly in real gas than in ideal gas.

II. The attenuation coefficient is higher in real gas than in ideal gas. With increase in frequency, the difference in attenuation coefficient between the ideal gas model and the real gas model becomes larger.

III. In downstream propagation, higher Mach number can reduce the influence of the real gas on phase velocity and attenuation coefficient of plane waves. In upstream case, the opposite is observed.

IV. For various radii of pipes, the effect of the real gas on phase velocity stays relatively steady. A larger radius of the pipe can strengthen the influence of the real gas on attenuation coefficient.

ACKNOWLEDGEMENT

This work was funded in part under Energy Innovation Research Programme (EIRP) Award NRF2014EWT-EIRP003-028, administrated by the Energy Market Authority (EMA). The EIRP is a competitive grant call initiative driven by the Energy Innovation Programme Office and funded by the National Research Foundation (NRF) of Singapore

REFERENCES

[1] Kirchhoff, Gustav. "Ueber den Einfluss der Wärmeleitung in einem Gase auf die Schallbewegung." *Annalen der Physik* 210.6 (1868): 177-193.

[2] Rayleigh, JW Strutt Baron. "The Theory of Sound, Vol. 2, revised and enlarged MacMillan." *London, New York* (1896): 132.

[3] Dokumaci, E. "Sound transmission in narrow pipes with superimposed uniform mean flow and acoustic modelling of automobile catalytic converters." *Journal of Sound and vibration*182.5 (1995): 799-808.

[4] Dokumaci, E. "On transmission of sound in circular and rectangular narrow pipes with superimposed mean flow." *Journal of sound and vibration* 210.3 (1998): 375-389.

[5] Y. Chen, Y. Huang, X. Chen, and D. Hu, "Axisymmetric wave propagation in uniform gas flow confined by rigid-walled pipeline." *Journal of Computational Acoustics* 22.04 (2014): 1450014.

[6] Y. Chen, X. Chen, Y. Huang, Y. Bai, D. Hu, and S. Fei, "Study of thermoviscous dissipation on axisymmetric wave propagating in a shear pipeline flow confined by rigid wall. Part I. theoretical formulation." *Acoustical Physics* 62.1 (2016).

[7] Y. Chen, X. Chen, Y. Huang, Y. Bai, D. Hu, and S. Fei, "Study of thermoviscous dissipation on axisymmetric wave propagating in a shear pipeline flow confined by rigid wall. Part II. Numerical study." *Acoustical Physics* 62.2 (2016).

[8] Y. Chen, Y. Huang, X. Chen, Y. Bai, and X. Tan, "Axisymmetric wave propagation in gas shear flow confined by a rigid-walled pipeline." *Chinese Physics B* 24.4 (2015): 044301.

[9] Jassim, Esam, M. Abedinzadegan Abdi, and Yuri Muzychka. "Computational fluid dynamics study for flow of natural gas through high-pressure supersonic nozzles: Part 1. Real gas effects and shockwave." *Petroleum Science and Technology* 26.15 (2008): 1757-1772.

[10] Jassim, Esam, M. Abedinzadegan Abdi, and Yuri Muzychka. "Computational fluid dynamics study for flow of natural gas through high-pressure supersonic nozzles: Part 2. Nozzle geometry and vorticity." *Petroleum Science and Technology* 26.15 (2008): 1773-1785.

[11] Chen, Ruey-Hung. *Foundations of Gas Dynamics*. Cambridge University Press, 2016.

[12] Fetter, Alexander L., and John Dirk Walecka. *Theoretical mechanics of particles and continua*. Courier Corporation, 2003.

[13] White, Frank M., and Isla Corfield. "Viscous fluid flow, vol. 3." (2006).

[14] Boccara, Nino. *Essentials of Mathematica: With Applications to Mathematics and Physics*. Springer Science & Business Media, 2007.