Age of Loop Oriented Wireless Networked Control System: Communication and Control Co-Design in the FBL Regime

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***Abstract*—In this paper, age of loop (AoL) is investigated for communication and control co-design in wireless networked control systems (WNCSs), to ensure information freshness in the sensor-controller-actuator loop with two-way delay. Also, the impact of fading channel and finite block length due to limited transmission bits are considered. A closed-form expression for Peak AoL (PAoL) is derived with respect to block length and the number of allowable transmissions in the presence of block error probability. It is shown that the average PAoL is less than the sum of the average peak Age of information (PAoI) in uplink (PAoU) and the PAoI in downlink (PAoD), due to the coupling of uplink and downlink. In addition, the average PAoL is proved to be convex with respect to block length and mono-decreasing with respect to the allowable transmission time. We then propose a low-complexity PAoL-aware block length adaptation scheme with retransmission policy and optimal control. Simulation results verify the correctness of our analytical results and demonstrate the effectiveness of the proposed PAoL-aware block length adaptation scheme in terms of information freshness and control performance.**

***Index Terms*—Age of loop, finite block length, communication and control co-design**

I. Introduction

Information freshness plays an important role in real-time applications, where the outdated packets with stale information may lead to erroneous operations [1]. To measure information freshness, age of information (AoI) was proposed and defined as the time elapsed since the generation of the latest successfully received status update [1]. Peak AoI (PAoI), which is an upper bound on AoI, was introduced to measure the worst case of AoI.

AoI and PAoI have been analyzed for wireless networked control systems (WNCSs) [2]–[6], where the interplay between AoI and sensing or control was explored. In [2] and [3], the estimation error was formulated as a function of age-penalty and AoI, respectively, where only the unreliable uplink (UL) was considered. Based on these, a WNCS with 2-way delay was considered in [4], and the optimal joint sensor/controller waiting policy in a Wiener-process system was proposed. In [5], the online transmission policy at the sensor was proposed to minimize remote estimation error based on the freshness-reliability tradeoff. Similar work has been investigated in [6], where the controller decides whether to sample a new packet or attempt for retransmission. In WNCSs, UL and DL are coupled, and the overall performance is affected by the packets of UL and DL in different manners. The aforementioned work analyzed the AoI in UL (AoU) or AoI in downlink (AoD) only, or assumed one perfect link in WNCSs, neglecting the impact of UL communication on the downlink (DL). In [7], the age of loop (AoL) and an AoL-based bandwidth allocation policy were proposed. However, the balance between UL and DL, and their relationship with overall AoL performance in WNCSs have not been studied.

In a typical real-time scenario in WNCSs, the status update information is of limited size and therefore transmitted with finite block length (FBL), *e.g.*, 20-250 bytes [8]. FBL is expected to reduce AoI, while it may induce inevitable block error probability (BLEP) and impair information freshness, due to limits on channel coding with FBL [9]. While most of the aforementioned work on WNCSs [2]–[7] have assumed infinite block length, which can not be applied in real-time WNCSs directly. In our previous work [11], the correlation between AoI and delay in the FBL regime was analyzed. In [12], block length adaptation was employed to minimize the average AoI for a single-hop FBL system. For WNCSs, the relationship between AoI and remote estimation error or control performance in the FBL regime was formulated in [13] and [14]. However, the impact of block length on the AoL of WNCSs in the FBL regime has not yet been analyzed, not to mention the block length adaptation for achieving the optimal overall performance in WNCSs.

Motivated by the open issues, we investigate peak AoL (PAoL) for WNCSs with FBL over fading channels. Our contributions are summarized as follows.

- To the best of our knowledge, this is the first work to investigate the average PAoL in a WNCS with two-way delay and BLEP in the FBL regime over fading channels.

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In contrast, the previous work on the AoI analysis of WNCSs has investigated UL and DL separately [2], [3], [5], [6], [12]–[14], or assumed IBL [2]–[7]. While in [13] and our previous work [11], only a single-hop system with FBL over additive white Gaussian noise (AWGN) channel was considered.

- To evaluate the information freshness of WNCSs, a closed-form expression for the average PAoL over fading channels is derived as a function of block length and the number of allowable transmission times in the presence of BLEP. Also, the relationship among the average PAoL, Peak AoU (PAoU) and peak AoD (PAoD) are analyzed, counterintuitively indicating that the average PAoL is less than the sum of PAoU and PAoD, due to the coupling of UL and DL. Our analytical results match the simulation results very well.
- To improve the performance of WNCSs, a PAoL-aware block length adaptation scheme with retransmission is proposed, and the corresponding optimal control is designed. Particularly, the number of allowable transmission times is determined first and then block lengths adaptations are conducted with low complexity. To obtain this, the monotonicity and convexity of the average PAoL with respect to the number of allowable transmission times and block lengths are analyzed. Simulation results show that the proposed scheme achieves a better average PAoL and control performance, compared to PAoU-aware and PAoD-aware schemes.

II. System Model

As depicted in Fig. 1, we pay attention to the linear time-invariant (LTI) system consists of the following parts [2].

- The sensor takes periodic samples of the plant state with period T and transmits them to the controller through an unreliable wireless channel, inducing delay τ_{u} and transmission error ϵ_{u} [17]. It is assumed that update packets contain l_{u} bits of information encoded into m_{u} symbols with the modulation order Θ and symbol duration T_{s}.
- After receiving the sensor data, the event-triggered controller calculates the control messages that contain l_{d} bits of information encoded into m_{d} symbols with the modulation order Θ and symbol duration T_{s} [17], [18], and transmits them to the actuator via wireless network, where the delay τ_{d} and transmission error ϵ_{d} are inevitable.
- The actuator acts based on the received control messages, and then the plant state is updated according to the control system dynamics.

A. Communication System in the FBL Regime

In this paper, channels are assumed to experience quasi-static Rayleigh fading, where the channel gain h remains constant during each transmission period and varies independently among different transmission periods [12], [15]. The instantaneous SNR is given by $\gamma = \bar{\gamma}\,|h|^2$, where $\bar{\gamma} = \frac{P_{\mathrm{t}}}{\sigma^2}$ denotes the average SNR with the transmit power P_{t} and average noise power σ^2. Also, it is assumed that the truncated

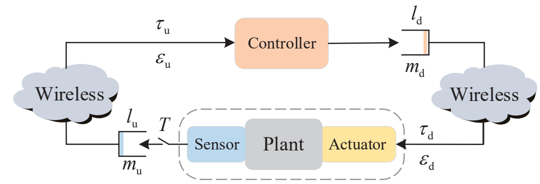

Fig. 1. System model of the single-loop WNCS.

automatic repeat request (TARQ) retransmission policy with low feedback overhead is adopted [12], [15], where the source keeps transmitting the current status update repeatedly until the allowable transmission times, r, is reached or the packet is transmitted successfully. Without loss of generality, we assume that the feedback message (acknowledgment/negative-acknowledgment) is transmitted perfectly without error [16], and that the trivial feedback delay is negligible [5].

For analytical tractability, the normal approximations of achievable rate and BLEP in the FBL regime are employed [9]. Therefore, the BLEP ϵ_α as a function of block length m_α, number of information bits l_α and SNR γ can be expressed as $\epsilon_\alpha \approx Q(\sqrt{m_\alpha/V(\gamma)}(C(\gamma)-l_\alpha/m_\alpha))$, which is an upper bound on the BLEP by neglecting the overhead of acquiring instantaneous SNR [10]. Note that the subscript α is adopted for the notion simplicity, where $\alpha = \mathrm{u}$ and d denote the UL and DL, respectively. Specifically, the Q-function is given by $Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}dt$. $C(\gamma) = \frac{1}{2}\log_2(1+\gamma)$ denotes the channel capacity per channel use (c.u.) and $V(\gamma) = \frac{1}{2}(1-\frac{1}{(1+\gamma)^2})(\log_2 e)^2$ denotes the channel dispersion. In the case of Rayleigh fading, the average BLEP is expressed as $\bar{\epsilon}_\alpha = \int_0^\infty f(\gamma)\epsilon_\alpha(\gamma)d\gamma$, where the probability density $f(\gamma)$ is an exponential function of SNR γ, as the channel gain h follows a complex Gaussian distribution with zero mean and unit variance. Then, the BLEP can be tightly approximated by transforming the Q-function into the form of segmented linear functions [12], *i.e.*,

$$\epsilon_\alpha(\gamma) \approx \begin{cases} 1, & \gamma < \delta_\alpha + \dfrac{1}{2\beta_\alpha} \\ \beta_\alpha(\gamma-\delta_\alpha)+1/2, & \delta_\alpha + \dfrac{1}{2\beta_\alpha} \leq \gamma < \delta_\alpha - \dfrac{1}{2\beta_\alpha} \\ 0, & \gamma > \delta_\alpha - \dfrac{1}{2\beta_\alpha}, \end{cases} \tag{1}$$

where $\delta_\alpha = e^{l_\alpha/m_\alpha}-1$ and $\beta_\alpha = -\sqrt{\frac{m_\alpha}{2\pi(e^{2l_\alpha/m_\alpha}-1)}}$. Based on (1), the average BLEP based on average SNR $\bar{\gamma}$ can be expressed as

$$\bar{\epsilon}_\alpha = \int_0^\infty \frac{1}{\bar{\gamma}} e^{-\frac{\gamma}{\bar{\gamma}}} \epsilon_\alpha(\gamma)\mathrm{d}\gamma \approx 1 + \beta_\alpha\bar{\gamma}(\mathrm{v}_{\alpha 1} - \mathrm{v}_{\alpha 2}), \tag{2}$$

where $v_{\alpha 1} = e^{-\frac{1}{\bar{\gamma}}(\delta_\alpha + \frac{1}{2\beta_\alpha})}$ and $v_{\alpha 2} = e^{-\frac{1}{\bar{\gamma}}(\delta_\alpha - \frac{1}{2\beta_\alpha})}$.

B. LTI Control System

The LTI system is modeled as [17]

$$d\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)dt + \mathbf{B}\mathbf{u}(t)dt + d\mathbf{w}(t), \tag{3}$$

where $\mathbf{x}(t)$ is the plant state, $\mathbf{u}(t)$ is the control input and $\mathbf{w}(t)$ is the AWGN with zero mean and variance $\mathbf{R}_w$. Also, $\mathbf{A}$ and $\mathbf{B}$ are the constant system-transition and control-input matrixes, respectively. As shown in Fig. 2, the plant state is sampled periodically by the sensor at times $t_n := nT, \forall n \in \mathcal{N}$. By considering zero-order-hold and letting subscript n denote the samples at t_n, we have [17], [18]

$$\mathbf{x}_{n+1} = e^{\mathbf{A}T}\mathbf{x}_n + \mathbf{\Gamma}_0^n \hat{\mathbf{u}}_n + \mathbf{\Gamma}_1^n \hat{\mathbf{u}}_{n-1} + \mathbf{w}_n, \tag{4}$$

where $\mathbf{\Gamma}_0^n = \int_0^{T-\tau_n} e^{\mathbf{A}t}\mathbf{B}dt$ and $\mathbf{\Gamma}_1^n = e^{\mathbf{A}(T-\tau_n)} \int_0^{\tau_n} e^{\mathbf{A}t}\mathbf{B}dt$ with the network delay $\tau_n = \tau_{\mathrm{u},n} + \tau_{\mathrm{d},n}$. Note that the delay in UL $\tau_{\mathrm{u},n}$ and DL $\tau_{\mathrm{d},n}$ depend on the block lengths and number of transmissions. Also, it is assumed that the network delay is bounded by one sampling period, *i.e.*, $\tau_n \leq T$, allowing packets to arrive at the receiver in the correct order [17], [18]. Let $p_\alpha = 1$ denote the event that the transmission in UL ($\alpha =$ u) or DL ($\alpha =$ d) is successful, whose probability is $1 - \epsilon_\alpha$. Also, we define $p_\alpha = 0$ as the event that the transmission is unsuccessful, with the probability of ϵ_α. Note that the actuator takes an action based on the previous control message in case of failed transmission [18]. Hence, the control input is given by $\hat{\mathbf{u}}_n = p_{\mathrm{u,n}} p_{\mathrm{d,n}} \mathbf{u}_n + (1 - p_{\mathrm{u,n}} p_{\mathrm{d,n}})\hat{\mathbf{u}}_{n-1}$, where $\mathbf{u}_n$ is designed in detail in Subsection IV-B.

III. Average PAoL Analysis for WNCSs

In this section, we present the closed-form expression for the average PAoL in the WNCS denoted by $\bar{A}^{\mathrm{L}}$. Also, the relationship among the average PAoL $\bar{A}^{\mathrm{L}}$, PAoU $\bar{A}^{\mathrm{U}}$ and PAoD $\bar{A}^{\mathrm{D}}$ is analyzed.

A. Average PAoL of WNCSs

In Fig. 2, the evolution of PAoL in the WNCS is shown. The instantaneous AoI measures the elapsed time since the generation of the last successfully received packet, *i.e.*, $\Delta(t) = t - u(t)$, where $u(t)$ is the generation time of the latest update that is successfully received by the receiver before time t. Assume that the i-th update is generated at time g_i by the source and received by the destination at time d_i. Then the service time of the i-th update can be expressed as $S_i = d_i - g_i$, and g_i^* denotes the packet that was not successfully received by the receiver. In addition, we denote the interarrival time by Y_i and the PAoL is denoted by $A_i = Y_i + S_i$. With the total number of successfully transmitted informative packets $N_\mathcal{T}$ over time interval $[0, \mathcal{T}]$, the average PAoL in the WNCS is given by

$$\begin{aligned}\bar{A} &= \lim_{\mathcal{T}\to\infty} \frac{1}{N_\mathcal{T}} \sum_{i=1}^{N_\mathcal{T}} A_i = \lim_{\mathcal{T}\to\infty} \frac{1}{N_\mathcal{T}} \sum_{i=1}^{N_\mathcal{T}} (S_i + Y_i) \\ &= \mathbb{E}[S] + \mathbb{E}[Y],\end{aligned} \tag{5}$$

where $\mathbb{E}[\cdot]$ denotes the expectation operator and the ergodicity of the stochastic process is assumed [1]. To obtain the average PAoL in (5), the average service time $\mathbb{E}[S]$ and interarrival time $\mathbb{E}[Y]$ are derived as follows.

The service time can be derived based on the transmission time of UL and DL under the event that the packets are successfully received by the destination. With the average BLEP in UL $\bar{\epsilon}_{\mathrm{u}}$ and DL $\bar{\epsilon}_{\mathrm{d}}$, as well as the number of allowable transmissions r, the average service time is derived in Appendix A and shown as

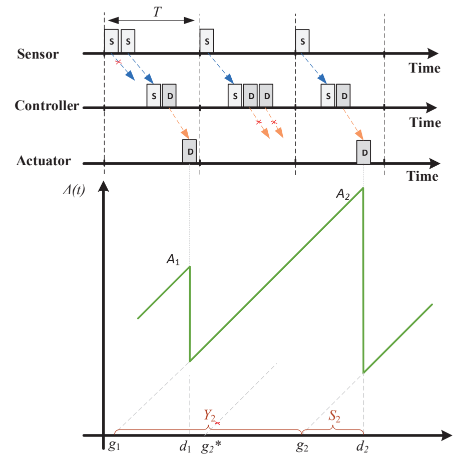

Fig. 2. Control timeline and the evolution of AoI in WNCSs.

$$\mathbb{E}[S] = \frac{m_{\mathrm{u}}T_{\mathrm{s}}}{1-\bar{\epsilon}_{\mathrm{u}}} - \frac{m_{\mathrm{u}}T_{\mathrm{s}}r\bar{\epsilon}_{\mathrm{u}}^r}{1-\bar{\epsilon}_{\mathrm{u}}^r} + \frac{m_{\mathrm{d}}T_{\mathrm{s}}}{1-\bar{\epsilon}_{\mathrm{d}}} - \frac{m_{\mathrm{d}}T_{\mathrm{s}}r\bar{\epsilon}_{\mathrm{d}}^r}{1-\bar{\epsilon}_{\mathrm{d}}^r}. \tag{6}$$

Since the instantaneous AoI at the actuator drops only if both packets in UL and DL are transmitted successfully, the interarrival time is given by

$$\mathbb{E}[Y] = \frac{T}{(1-\bar{\epsilon}_{\mathrm{u}}^r)(1-\bar{\epsilon}_{\mathrm{d}}^r)}. \tag{7}$$

Based on (6) and (7), the average PAoL in a WNCS is given in Theorem 1.

Theorem 1: *The average PAoL in a WNCS is formulated as a function of block lengths m_u and m_d, as well as the number of allowable transmissions r, given by*

$$\begin{aligned}\bar{A}^{\mathrm{L}} =& \frac{T}{(1-\bar{\epsilon}_{\mathrm{u}}^r)(1-\bar{\epsilon}_{\mathrm{d}}^r)} + \frac{m_{\mathrm{u}}T_{\mathrm{s}}}{1-\bar{\epsilon}_{\mathrm{u}}} - \frac{m_{\mathrm{u}}T_{\mathrm{s}}r\bar{\epsilon}_{\mathrm{u}}^r}{1-\bar{\epsilon}_{\mathrm{u}}^r} \\ &+ \frac{m_{\mathrm{d}}T_{\mathrm{s}}}{1-\bar{\epsilon}_{\mathrm{d}}} - \frac{m_{\mathrm{d}}T_{\mathrm{s}}r\bar{\epsilon}_{\mathrm{d}}^r}{1-\bar{\epsilon}_{\mathrm{d}}^r},\end{aligned} \tag{8}$$

Proof: Please refer to Appendix A. ■

Based on Theorem 1, the properties of the average PAoL are discussed in detail in Subsection IV-A.

B. PAoL vs. PAoU and PAoD

Based on Subsection III-A, the average PAoU and PAoD can be derived in a similar way and the relationship among PAoL, PAoU and PAoD is shown in Theorem 2.

Theorem 2: *The average PAoL is the sum of the average PAoU and PAoD with the average interarrival time in UL removed, given by*

$$\bar{A}^{\mathrm{L}} = \bar{A}^{\mathrm{D}} + \bar{A}^{\mathrm{U}} - \frac{T}{1-\bar{\epsilon}_{\mathrm{u}}^r}. \tag{9}$$

where $\bar{A}^{\mathrm{U}} = \frac{T}{1-\bar{\epsilon}_{\mathrm{u}}^r} + \frac{m_{\mathrm{u}}T_{\mathrm{s}}}{1-\bar{\epsilon}_{\mathrm{u}}} - \frac{m_{\mathrm{u}}T_{\mathrm{s}}r\bar{\epsilon}_{\mathrm{u}}^r}{1-\bar{\epsilon}_{\mathrm{u}}^r}$. and $\bar{A}^{\mathrm{D}} = \frac{T}{(1-\bar{\epsilon}_{\mathrm{u}}^r)(1-\bar{\epsilon}_{\mathrm{d}}^r)} + \frac{m_{\mathrm{d}}T_{\mathrm{s}}}{1-\bar{\epsilon}_{\mathrm{d}}} - \frac{m_{\mathrm{d}}T_{\mathrm{s}}r\bar{\epsilon}_{\mathrm{d}}^r}{1-\bar{\epsilon}_{\mathrm{d}}^r}$. ■

It is concluded from Theorem 2 that block lengths in UL and DL affect the average PAoL, PAoU and PAoD in different manners. Also, the average PAoD is mono-decreasing with respect to the block length m_{u}, as the BLEP $\bar{\epsilon}_{\mathrm{u}}$ is a mono-decreasing function of block length m_{u}. Then we have Remarks 1 and 2.

Remark 1: *Minimizing the average PAoU and PAoD is not equivalent to minimizing the average PAoL.*

Remark 2: *The average PAoU is independent of the block length m_d in DL. While the average PAoD is affected by both block lengths in UL and DL, and is mono-decreasing with respect to the block length m_u in UL.*

IV. PAoL-Aware Communication and Control Co-Design

This section aims to minimize the average PAoL in the WNCS by optimizing block lengths and the number of allowable transmissions. Based on these, the optimal feedback control is also designed.

A. PAoL-Aware Block Length Adaptation with Retransmission

We aim to minimize the average PAoL in a WNCS by jointly optimizing the block length vector $\mathbf{m} = (m_{\mathrm{u}}, m_{\mathrm{d}})$ and the number of allowable transmissions r. The problem is formulated as

$$\begin{aligned} \mathbf{P1}: \quad & \min_{\mathbf{m}, r} \quad \bar{A}^{\mathrm{L}} \\ & \text{s.t.} \quad (C1): l_\alpha / \log_2 \Theta \leq m_\alpha \leq m_{\alpha,\max}, m_\alpha \in \mathbb{N}_+, \\ & \qquad\ (C2): 1 \leq r \leq r_{\max}, \end{aligned} \tag{10}$$

where (C1) constraints the minimum block length that depends on the number of information bits l_α and modulation order Θ, and the maximum allowable block length $m_{\alpha,\max}$. (C2) limits the maximum allowable transmission time $r_{\max}$ that depends on the limits of power and sampling period.

First, we relax the integer constraint (C1) into continuous space, and then Propositions 1 and 2 are presented to address this optimization problem.

Proposition 1: *The average PAoL $\bar{A}^L$ is mono-decreasing with respect to the number of allowable transmissions r.*

Proof: The first-order derivative of the average PAoL with respect to the allowable transmissions is given by $\frac{\partial \bar{A}^{\mathrm{L}}}{\partial r} = \frac{T\bar{\epsilon}_{\mathrm{d}}^r \ln(\bar{\epsilon}_{\mathrm{d}})}{(1-\bar{\epsilon}_{\mathrm{d}}^r)^2(1-\bar{\epsilon}_{\mathrm{u}}^r)} + \frac{T\bar{\epsilon}_{\mathrm{u}}{}^r \ln(\bar{\epsilon}_{\mathrm{u}})}{(1-\bar{\epsilon}_{\mathrm{d}}^r)(1-\bar{\epsilon}_{\mathrm{u}}^r)^2} - \frac{m_{\mathrm{u}}T_{\mathrm{s}}\bar{\epsilon}_{\mathrm{u}}^r(\ln(\bar{\epsilon}_{\mathrm{u}})r-\bar{\epsilon}_{\mathrm{u}}^r+1)}{(1-\bar{\epsilon}_{\mathrm{u}}^r)^2} - \frac{m_{\mathrm{d}}T_{\mathrm{s}}\bar{\epsilon}_{\mathrm{d}}^r(\ln(\bar{\epsilon}_{\mathrm{d}})r-\bar{\epsilon}_{\mathrm{d}}^r+1)}{(1-\bar{\epsilon}_{\mathrm{d}}^r)^2} = \frac{T\bar{\epsilon}_{\mathrm{u}}^r \ln(\bar{\epsilon}_{\mathrm{u}})-m_{\mathrm{u}}T_{\mathrm{s}}\bar{\epsilon}_{\mathrm{u}}^r(\ln(\bar{\epsilon}_{\mathrm{u}})r-\bar{\epsilon}_{\mathrm{u}}^r+1)(1-\bar{\epsilon}_{\mathrm{d}}^r)}{(1-\bar{\epsilon}_{\mathrm{d}}^r)(1-\bar{\epsilon}_{\mathrm{u}}^r)^2} + \frac{T\bar{\epsilon}_{\mathrm{d}}^r \ln(\bar{\epsilon}_{\mathrm{d}})-m_{\mathrm{d}}\bar{\epsilon}_{\mathrm{d}}^r(\ln(\bar{\epsilon}_{\mathrm{d}})r-\bar{\epsilon}_{\mathrm{d}}^r+1)(1-\bar{\epsilon}_{\mathrm{u}}^r)}{(1-\bar{\epsilon}_{\mathrm{d}}^r)^2(1-\bar{\epsilon}_{\mathrm{u}}^r)}$. Since $T > r(m_{\mathrm{u}} + m_{\mathrm{d}})T_{\mathrm{s}}$, $T\bar{\epsilon}_{\mathrm{u}}^r \ln(\bar{\epsilon}_{\mathrm{u}}) - m_{\mathrm{u}}T_{\mathrm{s}}r\bar{\epsilon}_{\mathrm{u}}^r \ln(\bar{\epsilon}_{\mathrm{u}})(1 - \bar{\epsilon}_{\mathrm{d}}^r) - m_{\mathrm{u}}T_{\mathrm{s}}\bar{\epsilon}_{\mathrm{u}}^r(1 - \bar{\epsilon}_{\mathrm{u}}^r)(1 - \bar{\epsilon}_{\mathrm{d}}^r) < 0$ is held with $\bar{\epsilon}_{\mathrm{u}} < 1$ and $\bar{\epsilon}_{\mathrm{d}} < 1$. Similarly, $T\bar{\epsilon}_{\mathrm{d}}^r \ln(\bar{\epsilon}_{\mathrm{d}}) - m_{\mathrm{d}}T_{\mathrm{s}}\bar{\epsilon}_{\mathrm{d}}^r(\ln(\bar{\epsilon}_{\mathrm{d}})r - \bar{\epsilon}_{\mathrm{d}}^r + 1)(1 - \bar{\epsilon}_{\mathrm{u}}^r) < 0$ is guaranteed, and hence, $\frac{\partial \bar{A}^{\mathrm{L}}}{\partial r} < 0$ is proved. ■

Proposition 1 implies that retransmission is beneficial to reduce the average PAoL with the finite transmission time. The reason is that the probability of packets being successfully received can be increased by retransmission, which decreases the interarrival time and further the average PAoL.

Proposition 2: *When $l_\alpha \geq \pi$ and $\bar{\epsilon}_\alpha < 0.5$, the average PAoL $\bar{A}^L$ is convex with respect to block lengths* $\mathbf{m}$.

Proof: Please refer to Appendix B. ■

Proposition 2 implies that there exist optimal block lengths that minimize the average PAoL, based on which the block length adaptation is conducted.

It is concluded that **P1** is a convex optimization problem in its feasible domain based on Propositions 1 and 2. To handle this problem, we propose a PAoL-aware block length adaptation scheme with retransmission, where the optimal number of allowable transmission times $r^* = r_{\max}$ is determined first and then the optimal block lengths are obtained based on Newton's method [12] with low complexity, as shown in Algorithm 1.

Algorithm 1: Block Length Adaptation with Retransmission

1 **Input:** Maximum allowable transmission time $r_{\max}$, maximum allowable block length $m_{\alpha,\max}$, information bits l_α, initial point (m_{u_0}, m_{d_0}), error tolerance ξ.
2 The optimal number of allowable transmissions is given by $r^* = r_{\max}$.
3 Substitute r^* into (8) and obtain the optimal block lengths $\mathbf{m}^*$ based on Newton's method by rounding the optimized block lengths and comparing them with the block length requirements in (C1).
4 **Output:** Optimal block lengths $\mathbf{m}^*$ and allowable transmission time r^*.

Note that PAoU-aware and PAoD-aware policies can be obtained by a similar method and adopted as benchmarks, which are omitted due to limited space.

B. Optimal Feedback Control Design

Based on the optimized block lengths $\mathbf{m}^*$ and number of allowable transmissions r^* derived in Subsection IV-A, the corresponding BLEP can be obtained as ϵ_α^*. Also, τ_n and $\hat{\mathbf{u}}_n$ can be updated based on Subsection II-B. Then, we introduce an augmented discrete-time state variable $\mathbf{z}_n = [\mathbf{x}_n \quad \hat{\mathbf{u}}_{n-1}]^T$ to analyze the system, which can be further expressed as

$$\mathbf{z}_{n+1} = \mathbf{\Phi}(T, \tau_n)\mathbf{z}_n + \mathbf{\Gamma}(T, \tau_n)\hat{\mathbf{u}}_n + \mathbf{w}_n, \tag{11}$$

where $\mathbf{\Phi}(T, \tau_n) = \begin{bmatrix} e^{\mathbf{A}T} & \mathbf{\Gamma}_1^n \\ 0 & 0 \end{bmatrix}$ and $\mathbf{\Gamma}(T, \tau_n) = \begin{bmatrix} \mathbf{\Gamma}_0^n \\ I \end{bmatrix}$.

Particularly, the control performance is considered and defined as the reciprocal of control cost, which is given by [17]

$$J = \mathbb{E}\{\sum_{n=0}^{N-1}[\mathbf{z}_n^\top \mathbf{Q}\mathbf{z}_n + \hat{\mathbf{u}}_n^\top R\hat{\mathbf{u}}_n] + \mathbf{z}_N^\top \mathbf{Q}\mathbf{z}_N\}, \tag{12}$$

where $\mathbf{Q}$ and R are positive defined weight factors. Then, the optimal control law is designed to minimize the control cost, given by $\mathbf{u}_n = -\mathbf{K}\mathbf{z}_n$ [18], where $\mathbf{K}$ is the control gain given by $\mathbf{K} = (R + \mathbf{\Gamma}^\top\mathbf{P}\mathbf{\Gamma})^{-1}\mathbf{\Gamma}^\top\mathbf{P}\mathbf{\Phi}$ and $\mathbf{P}$ is the steady-state solution of the control Riccati recursion $\mathbf{P} = \mathbf{\Phi}^\top\mathbf{P}\mathbf{\Phi} - (\mathbf{\Phi}^\top\mathbf{P}\mathbf{\Gamma})(\mathbf{\Gamma}^\top\mathbf{P}\mathbf{\Gamma} + R)^{-1}(\mathbf{\Gamma}^\top\mathbf{P}\mathbf{\Phi}) + \boldsymbol{Q}$ [17].

V. Simulation Results

In this section, we first show the impact of block lengths and the number of allowable transmissions on the average PAoL in WNCSs. Then, the performance of the proposed PAoL-aware communication and control co-design is demonstrated via simulation results.

Following [8] and [17], we consider a typical WNCS in factory automation, where an unstable and controllable plant in the form of (1) is formulated with $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\boldsymbol{Q} = I$ and $R = 0.1$, where I is an identity matrix. The sampling period of the sensor is set to $T = 0.01$ s and the symbol duration is set to $T_{\mathrm{s}} = 10^{-5}$ s [11]. Both the sensor and controller are allowed to transmit with the number of allowable transmissions $r = 2$ [15]. The number of information bits is set to $l_{\mathrm{u}} = 240$ and $l_{\mathrm{d}} = 160$ at the sensor and controller [8], respectively, with the maximum allowable block lengths $m_{\mathrm{u,max}} = 400$ c.u. and $m_{\mathrm{u,max}} = 300$ c.u., and the modulation order $\Theta = 4$. In addition, the initial point of the proposed algorithm is set to (60, 40) with the error tolerance $\xi = 10^{-8}$.

Fig. 3 depicts the joint impact of block lengths in UL and DL on the average PAoL with the sampling period $T = 10$ ms, number of allowable transmissions $r = 2$ and average SNR $\bar{\gamma} = 6$ dB. It is observed that there exists an optimal block length pair that minimizes the average PAoL, which can be found by using the proposed PAoL-aware transmission policy. Its optimality is validated by using the exhaustive search.

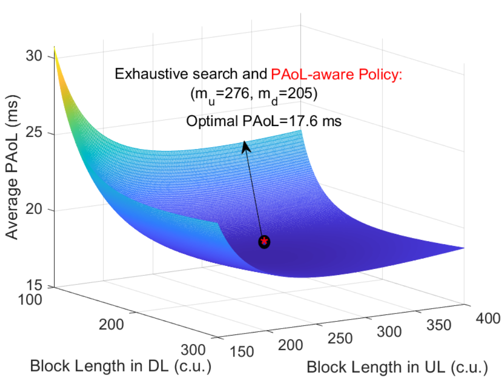

Fig. 3. Joint impact of block lengths on the average PAoL with sampling period $T = 10$ ms, numer of allowable transmissions $r = 2$ and average SNR $\bar{\gamma} = 6$ dB.

Fig. 4 shows the impact of the DL (UL) block length with the fixed UL (DL) block length on the average PAoU, PAoD and PAoL, respectively. Interestingly, with the fixed UL block length $m_{\mathrm{u}} = 200$ c.u., both the average PAoD and PAoL are convex with respect to the block length in DL, while the average PAoU is independent of it, as analyzed in Theorem 1. In addition, the block length in UL affects the average PAoU, PAoD and PAoL, even with the fixed block length in DL $m_{\mathrm{d}} = 100$ c.u.. Also, the average PAoD is mono-decreasing with respect to the block length in UL, as stated in Remark 2. This is due to that a larger block length m_{u} leads to a lower BLEP and hence, the interarrival time at the controller is smaller, which leads to a lower average PAoD. Furthermore, it is shown that the average PAoU, PAoD and PAoL have different trends with respect to block lengths, and hence, the average PAoL is not the sum of the average PAoU and PAoD, as stated in Theorem 2 and Remark 1.

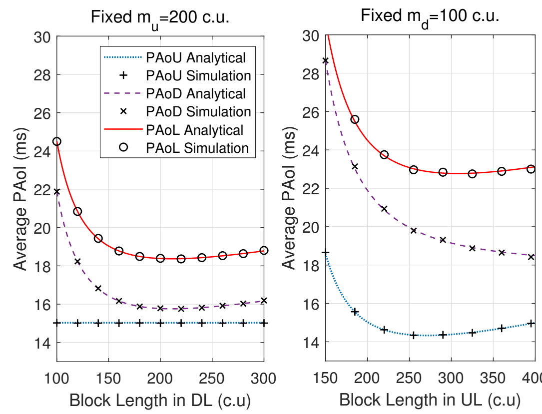

Fig. 4. Impact of the block length in DL (UL) with the fixed block length in UL (DL) on the average PAoU, PAoD and PAoL, respectively, with the sampling period $T = 10$ ms, number of allowable transmissions $r = 2$ and average SNR $\bar{\gamma} = 6$ dB.

Fig. 5 compares the proposed PAoL-aware policy and the existing approaches in terms of average PAoL and control performance, where the control performance is defined as the reciprocal of the control cost, *i.e.*, $\frac{1}{J}$. Generally, with a higher SNR, the average PAoL is lower and the control performance is better. Also, it is shown that the PAoL-aware policy achieves the lowest average PAoL and the best control performance, which outperforms the approach without retransmission, especially with low SNRs. Furthermore, the proposed PAoL-aware policy outperforms the approaches that optimize PAoU and PAoD [12] solely, where the improvements of the average PAoL and control performance are up to 34% and 6%, respectively, demonstrating the advantages of the introduced average PAoL.

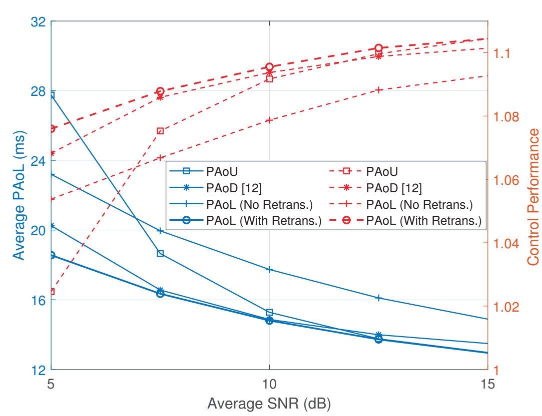

Fig. 5. Comparison between different transmission policies.

VI. Conclusion

In this paper, we have presented a comprehensive analysis on the average PAoL and the co-design of communication and control in the WNCS with FBL over fading channels. The closed-form expressions for the average PAoU, PAoD and PAoL with respect to block lengths and allowable transmission times have been derived. The accuracy of the theoretical analysis has been confirmed by simulations. Based on these, the average PAoL has been proved to be less than the sum of PAoU and PAoD. Particularly, the PAoL-aware communication and control co-design has been investigated. Particularly, a low-complexity PAoL-aware block length adaptation with retransmission policy has been proposed, where the optimal allowable transmission times and block lengths are derived successively. The proposed PAoL-aware communication policy with optimal control has provided a up to 34% PAoL reduction and 6% control performance improvement, compared to the approaches that optimize PAoU and PAoD solely.

Appendix A
Proof of Theorem 1

Since we consider TARQ at the sensor and controller, the transmission delay of each round is given by $m_\mathrm{u}T_\mathrm{s}$ and $m_\mathrm{d}T_\mathrm{s}$, respectively. Then, with the allowable transmission time r and the case of successful transmission, the average service time is derived as $\mathbb{E}[S] = \frac{\sum_{k=1}^{r} k\bar{\epsilon}_\mathrm{u}^{k-1}(1-\bar{\epsilon}_\mathrm{u})m_\mathrm{u}T_\mathrm{s}}{1-\bar{\epsilon}_\mathrm{u}^r} + \frac{\sum_{k=1}^{r} k\bar{\epsilon}_\mathrm{d}^{k-1}(1-\bar{\epsilon}_\mathrm{d})m_\mathrm{d}T_\mathrm{s}}{1-\bar{\epsilon}_\mathrm{d}^r}$, as the BLEP of different status packets is independent and the PAoL characterizes the timeliness of packets generated from the sensor and received by the actuator. Based on the series expansion [14], we have $\mathbb{E}[S] = \frac{m_\mathrm{u}T_\mathrm{s}}{1-\bar{\epsilon}_\mathrm{u}} - \frac{r\bar{\epsilon}_\mathrm{u}^r m_\mathrm{u}T_\mathrm{s}}{1-\bar{\epsilon}_\mathrm{u}^r} + \frac{m_\mathrm{d}T_\mathrm{s}}{1-\bar{\epsilon}_\mathrm{d}} - \frac{r\bar{\epsilon}_\mathrm{d}^r m_\mathrm{d}T_\mathrm{s}}{1-\bar{\epsilon}_\mathrm{d}^r}$, as shown in (6). Considering the BLEP $\bar{\epsilon}_\mathrm{u}$ in UL and $\bar{\epsilon}_\mathrm{d}$ in DL, and letting $x = 1-(1-\bar{\epsilon}_\mathrm{u}^r)(1-\bar{\epsilon}_\mathrm{d}^r)$, the average interarrival time of informative packets is given by $\mathbb{E}[Y] = \sum_{k=0}^{\infty}(k+1)x^k T(1-x)$, based on the fact that AoL drops only when both packets in UL and DL are transmitted successfully. Applying the series $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ and $\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$, we have $\mathbb{E}[Y] = \frac{T}{(1-\bar{\epsilon}_\mathrm{u}^r)(1-\bar{\epsilon}_\mathrm{d}^r)}$, yielding the average PAoL in (8).

Appendix B
Proof of Proposition 2

The second-order derivative of the average PAoL with respect to the block length in UL is given by $\frac{\partial^2 \bar{A}^\mathrm{L}}{\partial {m_\mathrm{u}}^2} = F_1 + F_2 + F_3$, where $F_1 = \frac{\frac{T}{1-\bar{\epsilon}_\mathrm{d}^r}\frac{\partial^2 \bar{\epsilon}_\mathrm{u}^r}{\partial {m_\mathrm{u}}^2}(1-\bar{\epsilon}_\mathrm{u}^r) + \frac{2T}{1-\bar{\epsilon}_\mathrm{d}^r}(\frac{\partial \bar{\epsilon}_\mathrm{u}^r}{\partial m_\mathrm{u}})^2}{(1-\bar{\epsilon}_\mathrm{u}^r)^3}$, $F_2 = \frac{m_\mathrm{u}T_\mathrm{s}\frac{\partial^2 \bar{\epsilon}_\mathrm{u}}{\partial m_\mathrm{u}^2}(1-\bar{\epsilon}_\mathrm{u}) + 2T_\mathrm{s}(1-\bar{\epsilon}_\mathrm{u})\frac{\partial \bar{\epsilon}_\mathrm{u}}{\partial m_\mathrm{u}} + 2m_\mathrm{u}T_\mathrm{s}(\frac{\partial \bar{\epsilon}_\mathrm{u}}{\partial m_\mathrm{u}})^2}{(1-\bar{\epsilon}_\mathrm{u})^3}$ and $F_3 = \frac{m_\mathrm{u}T_\mathrm{s}r\frac{\partial^2 \bar{\epsilon}_\mathrm{u}^r}{\partial m_\mathrm{u}^2}(1-\bar{\epsilon}_\mathrm{u}^r) + 2T_\mathrm{s}r(1-\bar{\epsilon}_\mathrm{u}^r)\frac{\partial \bar{\epsilon}_\mathrm{u}^r}{\partial m_\mathrm{u}} + 2m_\mathrm{u}T_\mathrm{s}r(\frac{\partial \bar{\epsilon}_\mathrm{u}^r}{\partial m_\mathrm{u}})^2}{(1-\bar{\epsilon}_\mathrm{u}^r)^3}$. Then, we introduce auxiliary variables $F_4 = (\frac{2T}{1-\bar{\epsilon}_\mathrm{d}^r} - 2mT_\mathrm{s}r)(\frac{\partial \bar{\epsilon}_\mathrm{u}^r}{\partial m_\mathrm{u}})^2 + 2m_\mathrm{u}T_\mathrm{s}(\frac{\partial \bar{\epsilon}_\mathrm{u}}{\partial m_\mathrm{u}})^2$, $F_5 = (\frac{T}{1-\bar{\epsilon}_\mathrm{d}^r} - m_\mathrm{u}T_s r)\frac{\partial \bar{\epsilon}_\mathrm{u}^r}{\partial m_\mathrm{u}^2} + m_\mathrm{u}T_\mathrm{s}\frac{\partial^2 \bar{\epsilon}_\mathrm{u}}{\partial m_\mathrm{u}^2}$ and $F_6 = -2T_\mathrm{s}r(1-\bar{\epsilon}_\mathrm{u}^r)\frac{\partial \bar{\epsilon}_\mathrm{u}^r}{\partial m_\mathrm{u}} + 2T_\mathrm{s}(1-\bar{\epsilon}_\mathrm{u})\frac{\partial \bar{\epsilon}_\mathrm{u}}{\partial m_\mathrm{u}}$. Since $0 < \bar{\epsilon}_\mathrm{u} < 1$, $F_1 + F_2 + F_3 > 0$ is guaranteed by proving $F_4 + F_5 + F_6 > 0$. First, since $\frac{T}{1-\bar{\epsilon}_\mathrm{d}^r} - m_\mathrm{u}T_s r > 0$ is held, $F_4 > 0$ is proved. Then, F_5 can be expressed as $F_5 = ((\frac{T}{1-\bar{\epsilon}_\mathrm{d}^r} - m_\mathrm{u}T_s r)r! + m_\mathrm{u}T_\mathrm{s})\frac{\partial^2 \bar{\epsilon}_\mathrm{u}}{\partial m_\mathrm{u}^2}$. Based on the first-order Taylor approximation of $\bar{\epsilon}_\mathrm{u}$ [12], we have $\frac{\partial^2 \bar{\epsilon}_\mathrm{u}}{\partial m_\mathrm{u}^2} \approx \frac{(2l_\mathrm{u}m_\mathrm{u}+l_\mathrm{u}^2)e^{l_\mathrm{u}/m_\mathrm{u}} - 2\sqrt{\pi l_\mathrm{u}}m_\mathrm{u}}{m_\mathrm{u}^4\bar{\gamma}} > 0$, if $l_\mathrm{u} \geq \pi$, which is easily satisfied for typical scenarios [12]. Then, $F_6 > 0$ holds as $\frac{\partial \bar{\epsilon}_\mathrm{u}}{\partial m_\mathrm{u}} = \frac{(\sqrt{\pi l_\mathrm{u}} - l_\mathrm{u}e^{l_\mathrm{u}/m_\mathrm{u}})}{\bar{\gamma}m_\mathrm{u}^2} e^{\frac{-e^{l_\mathrm{u}/m_\mathrm{u}} + \frac{\sqrt{\pi l_\mathrm{u}}}{m_\mathrm{u}} + 1}{\bar{\gamma}}} < 0$. Then, $\frac{\partial^2 \bar{A}^\mathrm{L}}{\partial {m_\mathrm{d}}^2} > 0$ can be proved in a similar way. In addition, we have $\frac{\partial \bar{A}^\mathrm{L}}{\partial m_\mathrm{u}\partial m_\mathrm{d}} = \frac{T\frac{\partial \bar{\epsilon}_\mathrm{u}^r}{\partial m_\mathrm{u}}\frac{\partial \bar{\epsilon}_\mathrm{d}^r}{\partial m_\mathrm{d}}}{(1-\bar{\epsilon}_\mathrm{d}^r)^2(1-\bar{\epsilon}_\mathrm{u}^r)^2}$ and $F_1 = \frac{T\frac{\partial^2 \bar{\epsilon}_\mathrm{u}^r}{\partial {m_\mathrm{u}}^2}(1-\bar{\epsilon}_\mathrm{d}^r) + 2T\frac{1-\bar{\epsilon}_\mathrm{d}^r}{1-\bar{\epsilon}_\mathrm{u}^r}(\frac{\partial \bar{\epsilon}_\mathrm{u}^r}{\partial m_\mathrm{u}})^2}{(1-\bar{\epsilon}_\mathrm{u}^r)^2(1-\bar{\epsilon}_\mathrm{d}^r)^2}$. Then, $\frac{\partial^2 \bar{A}^\mathrm{L}}{\partial {m_\mathrm{u}}^2}\frac{\partial^2 \bar{A}^\mathrm{L}}{\partial m_\mathrm{d}^2} - \frac{\partial \bar{A}^\mathrm{L}}{\partial m_\mathrm{u}\partial m_\mathrm{d}}\frac{\partial \bar{A}^\mathrm{L}}{\partial m_\mathrm{d}\partial m_\mathrm{u}} > 0$ when $\bar{\epsilon}_\mathrm{d}^r - \bar{\epsilon}_\mathrm{u}^r < 0.5$, which can be guaranteed as $0 < \bar{\epsilon}_\alpha < 0.5$ is held for time-critical applications [11]. Hence, the Hessian matrix of the objective function with respect to block lengths is positive definite.

References

[1] S. Kaul, R. Yates, and M. Gruteser, "Real-time status: How often should one update?" in *Proc. IEEE INFOCOM*, Orlando, FL, USA, Mar. 2012, pp. 2731–2735.

[2] M. Klügel *et al.*, "AoI-penalty minimization for networked control systems with packet loss," in *Proc. IEEE INFOCOM Workshops*, Paris, France, 2019, pp. 189-196.

[3] J. P. Champati *et al.*, "Performance characterization using AoI in a single-loop networked control system," in *Proc. IEEE INFOCOM Workshops*, Paris, France, 2019, pp. 197-203.

[4] C.-H. Tsai *et al.*, "Unifying AoI minimization and remote estimation-optimal sensor/controller coordination with random two-way delay," in *Proc. IEEE INFOCOM*, Toronto, ON, Canada, 2020, pp. 466-475

[5] K. Huang *et al.*, "Real-time remote estimation with hybrid ARQ in wireless networked control," *IEEE Trans. Commun.*, vol. 19, no. 5, pp. 3490-3504, May 2020.

[6] E. Fountoulakis *et al.*, "Optimal sampling cost in wireless networks with age of information constraints," in *Proc. IEEE INFOCOM Workshops*, Toronto, ON, Canada, 2020, pp. 918-923.

[7] P. M. de Sant Ana *et al.*, "Age of loop for wireless networked control systems optimization," in *Proc. IEEE PIMRC*, Helsinki, Finland, 2021, pp. 1-7.

[8] V. 3GPP, TS 22.104, "Service requirements for cyber-physical control applications in vertical domains; stage-1 (release 17)," Jun. 2019.

[9] Y. Polyanskiy, H. V. Poor, and S. Verdú, "Channel coding rate in the finite blocklength regime," *IEEE Trans. Inf. Theory*, vol. 56, no. 5, pp. 2307–2359, May 2010.

[10] J. Cao et al., "Independent Pilots vs. Shared Pilot: Short Frame Structure Optimization for Heterogeneous-Traffic URLLC Networks," *IEEE Trans. Wirel. Commun.*, Early Access 2022.

[11] J. Cao *et al.*, "Information age-delay correlation and optimization With finite block length," *IEEE Trans. Commun.*, vol. 69, no. 11, pp. 7236-7250, Nov. 2021.

[12] B. Yu *et al.*, "Average age of information in short-packet-based machine type communication," *IEEE Trans. Veh. Technol.*, vol. 69, no. 9, pp. 10306-10319, Sep. 2020.

[13] S. Roth *et al.*, "Remote short blocklength process monitoring: trade-off between resolution and data freshness," in *Proc. IEEE ICC*, Dublin, Ireland, 2020, pp. 1-6.

[14] X. Wang *et al.*, "AoI-aware control and communication co-design for industrial IoT systems," *IEEE Internet of Things J.*, vol. 8, no. 10, pp. 8464-8473, May, 2021.

[15] Y. Gu *et al.*, "Timely status update in internet of things monitoring systems: An age-energy tradeoff," *IEEE Internet of Things J.* , vol. 6, no. 3, pp. 5324-5335, Jun. 2019.

[16] A. Arafa *et al.*, "Timely Transmissions Using Optimized Variable Length Coding," in *Proc. IEEE INFOCOM Workshops*, Vancouver, BC, Canada, May 2021, pp. 1-6.

[17] P. Par k*et al.*, "Wireless networked control system co-design," in *Proc. IEEE ICNSC*, Delft, Netherlands, Apr. 2011, pp. 486-491.

[18] J. P. Hespanha *et al.*, "A Survey of Recent Results in Networked Control Systems," *Proc. of the IEEE*, vol. 95, no. 1, pp. 138-162, Jan. 2007.