Frame-Based Compressive Sensing MR Image Reconstruction with Balanced Regularization

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***Abstract*— This paper addresses the frame-based MR image reconstruction from undersampled k-space measurements by using a balanced ℓ_1-regularized approach. Analysis-based and synthesis-based approaches are two common methods in ℓ_1-regularized image restoration. They are equivalent under the orthogonal transform, but there exists a gap between them under redundant transform such as frame. Thus the third approach was developed to reduce the gap by penalizing the distance between the representation vector and the canonical frame coefficient of the estimated image, this balanced approach bridges the synthesis-based and analysis-based approaches and balances the fidelity, sparsity and smoothness of the solution. These frame-based approaches have been studied and compared for optical image restoration over the last few years. In this paper, we further study and compare these three approaches for the compressed sensing MR image reconstruction under redundant frame domain. These ℓ_1-regularized optimization problems are solved by using a variable splitting strategy and the classical alternating direction method of multiplier (ADMM). Numerical simulation results show that the balanced approach can reduce the gap between the analysis-based and synthesis-based approaches and are even better than these two approaches under our experimental conditions.**

I. Introduction

Magnetic resonance imaging (MRI) has been becoming important and popular clinical imaging equipment in disease diagnosis and treatment over the past decades because it can provide non-invasive in vivo images of soft tissue change. But due to physical and physiological limitations, most MRI scanners take long scanning time to acquire a sequence of data and reconstruct an MR image. Therefore, reducing the scanning time is necessary and well-motivated problem for MR imaging.

Recent studies in compressive sensing theory and its applications to MRI ([1]– [2]) indicate that MR images can be accurately reconstructed from highly undersampled *k*-space data by employing nonlinear reconstruction schemes. More specifically, if the underlying image exhibits transformed sparsity, and *k*-space undersampling results in incoherent artifacts in the transform domain, then the image can be recovered from randomly undersampled frequency domain data by using an appropriate nonlinear recovery scheme. That is to say, CS-based MRI has the potential to reduce the scanning time considerably by *k*-space data undersampling while keeping the high quality of MRI images.

It can be seen that image reconstruction from undersampled *k*-space measurements is a crucial task in the CS-based MR imaging system, and it can be formulated as solving an ℓ_1-regularized optimization problem [1]– [2]. Under different prior assumptions of signal sparsity, the variety of published work in this topic can be classified into two groups: synthesis-based approach and analysis-based approach. The former has ℓ_1-regularizer that penalizes the transforms of the image and analyzes the image itself, so it is called analysis-based approach. While the latter is called synthesis-based approach since it penalizes the sparse representation vector in terms of the atom signals (the columns of some transforms or dictionaries), and synthesizes the image from the sparsest representation vector. In orthogonal transform or invertible dictionary, these two groups are equivalent, but in redundant ones, there exists gap between them [4]. For frame-based image restoration, an alternative approach has been studied in order to balance these two approaches, so it is called balanced approach [8], [12]. In balanced approach, another term is added in the synthesis-based minimization problem that penalizes the distance between the sparse representation vector and the frame coefficient of the estimated image. Thus the balanced approach bridges the analysis-based approach and synthesis-based approach and reduces the gap between them. Moreover, these three approaches are developed independently in the literature, and it is very difficult to say which is better since each has its own favorable data sets and applications. While the analysis-based and synthesis-based approaches are often used in image restoration, the balanced approach that balances these two approaches can be a useful choice in the frame-based image restoration such as image deblurring/deconvolution and image inpainting [12].

Although both analysis-based and synthesis-based approaches are compared in natural image restoration that captured by optical cameras, they have not been compared in frame-based MR image reconstruction. Moreover, the third balanced approach has not been investigated in CS-based MR image reconstruction. In this paper, it is the first time to apply the balanced approach into frame-based MRI reconstruction and compare it with commonly used analysis-based and synthesis-based approaches. Experimental results in this paper show that there exists gap between analysis-based and synthesis-based approaches in frame-based MR image reconstruction, the balanced approach can bridge the gap between the synthesis-based and analysis-based approaches and can also provide better performance in our experimental case studies. But similar to the natural optical image restoration, it is hard to draw a definitive conclusion as to which is the better in all CS-MR image reconstruction cases since each has its own favorable data sets and applications.

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These reconstruction optimization problems are solved by combining a variable splitting and the classical alternating direction method of multipliers (ADMM) in this paper. ADMM is very simple in concepts and suitable for large-scale systems [10], [11]. Moreover, since MRI scanners acquire data in the spatial-frequency encoded domain, the regularized Hessian matrix and its inverse perform quite efficiently by exploiting fast Fourier transform and fast tight wavelet frame algorithms. Thus the second-order information in Hessian matrix can be used accurately, which speeds up the convergence of the proposed algorithms compared with other first-order gradient algorithms [9], [12].

II. Problem Formulation

The objective of compressed sensing MR image reconstruction is to recover the unknown true MR image $u \in \mathbb{R}^n$ from a noisy undersampled k-space measurement $y \in \mathbb{C}^m$ ($m \ll n$) modeled by

$$y = Bu + \nu \tag{1}$$

where $B = SF$ is linear sampling operator with $F \in \mathbb{C}^{n\times n}$ being discrete Fourier transform and $S \in \mathbb{R}^{m\times n}$ sampling pattern containing m rows of identity matrix of order n, $\nu \in \mathbb{C}^m$ is assumed to be a circular complex white Gaussian noise with mean zero and variance σ^2.

It is well known that MR image reconstruction is an ill-posed inverse problem due to noise and undersampling. But in order to obtain acceptable solution, variational regularization approaches are often used to find approximate ones by exploiting signal prior information. Recently the sparse prior of signal under a certain orthogonal transform or redundant frame has been exploited efficiently [3], [4].

A signal u is said to have a sparse representation over a frame $W \in \mathbb{R}^{n\times d}$, if there exists a sparse vector $x \in \mathbb{R}^d$ such that $u = Wx$. In general, the frame is redundant [5], [6]. In this paper, the redundant and normalized tight frame (Parseval frame) is used, i.e., $WW^T = I$, where I denotes the identity matrix. Thus $u = W(W^T u)$ for every vector $u \in \mathbb{R}^n$, and W is called synthesis operator and W^T is called analysis operator. The components of the vector $W^T u$ are called the canonical frame coefficients representing u. It is known that MR images have sparse representations under wavelet-like frame, so the wavelet-like frame sparsity is exploited as the prior information in this paper.

In view of sparse representation of MR image, the frame-based MR image reconstruction can be described as: the sparse representation vector x is estimated from the noisy image under the sparsity assumption first, then the unknown image u can be constructed as a linear combination of a few columns of frame W. The corresponding ℓ_1-regularized minimization problem is

$$\min_x \frac{1}{2}\|BWx - y\|_2^2 + \lambda^T |x|_1 \tag{2}$$

where $\|\cdot\|_2$ denotes the l_2-norm and $|z|_1$ denotes the vector obtained from z by taking absolute values of its elements, λ is given nonnegative weight vector. This is called the synthesis-based approach, since only the sparsity of representation vector is penalized and the estimated image is synthesized by the sparsest representation vector.

Another method is to estimate the unknown image directly by solving a minimization problem as

$$\min_{u\in\mathbb{R}^n} \frac{1}{2}\|Bu - y\|_2^2 + \lambda^T |W^T u|_1. \tag{3}$$

This is called analysis-based approach because the canonical frame coefficient of the analysis operator W^T is penalized. It is noted that in (3) only the sparsity of the canonical frame coefficient vector is penalized, which corresponds to the smoothness of the underlying image [7].

Both analysis-based and synthesis-based approaches are often used in the literature, in particular, they are equivalent under orthogonal transforms. But the tight wavelet frame systems are redundant, the mapping from the image u to its coefficients is not one-to-one, i.e., the representation of u in the frame domain is not unique. Moreover, for tight frame systems, $WW^T = I$, but $W^T W \neq I$. Therefore, normally the estimated representation vector x by synthesis-based approach is not equal to the canonical frame coefficient vector of its reconstructed image $\hat{u} = Wx$, i.e., $x \neq W^T(Wx)$. Thus there exists a gap between the estimated images using synthesis-based approach and analysis-based approach [4]. In order to reduce the gap, it is natural to add another term to penalize the difference $x - W^T(Wx)$ in the optimization problem. Then the balanced ℓ_1-regularized approach is formulated as

$$\min_x \frac{1}{2}\|BWx - y\|_2^2 + \frac{\gamma}{2}\|(I - W^T W)x\|_2^2 + \lambda^T |x|_1 \tag{4}$$

where γ and λ are given nonnegative weight vectors. The first term denotes penalty on the data fidelity, the last term penalizes the sparsity of coefficient vector, the second term penalizes the distance between the frame representation vector x and the canonical frame coefficient vector of synthesized image $\hat{u} = Wx$. The larger γ is, the closer x to the canonical frame coefficients of $\hat{u}$ is.

It should be noted that when $\gamma = 0$, the problem (4) is reduced to synthesis-based approach, while on the other extreme, when $\gamma = \infty$, the term $\|(I - W^T W)x\|_2^2$ must be 0 if the problem (4) has a finite solution. i.e., $x = W^T u$ for some $u \in \mathbb{R}^n$. Thus the problem (4) is reduced to analysis-based approach. Obviously, the balanced approach provides an unified form for both synthesis-based and analysis-based approaches. It can also be seen that the problem (4) balances the sparsity of the frame representation vector and the smoothness of the image, hence it is called the balanced approach.

It is also noted that when the columns of W form an orthonormal basis, the above three approaches are exactly the same. However, for the redundant tight frame W, these three approaches can not be derived from one another. It is hard to draw definitive conclusions as to which approach is better since each has its own favorable data sets and applications.

The balanced approach has been studied in [7], [8], [12] for the optical image restoration such as deblurring and inpainting, but it is still not be applied to compressed sensing MR image reconstruction. In this paper, we shall study and compare it with analysis-based and synthesis-based approaches in frame-based MR image reconstruction from undersampled measurements.

III. ADMM-based Algorithms

In this section, we apply variable splitting technique and ADMM algorithm to solve the balanced regularization problem (4) for frame-based MR image reconstruction from undersampled measurements. Using variable splitting, this problem can be rewritten as an equivalent constrained optimization problem:

$$\min_{x,v\in R^n} \frac{1}{2}\|BWx-y\|_2^2+\frac{\gamma}{2}\|(I-W^TW)x\|_2^2+\lambda^T|v|_1 \quad \text{subject to } x=v. \tag{5}$$

Then along the lines of [12], and using classical ADMM technique to (5), the balanced regularization optimization problem can be solved by the following algorithm.

Algorithm: ADMM for balanced regularization (5):

1) Set $k=0$, choose $\mu>0$, v_0 and d_0.
2) *repeat*
3) $r_k=W^TB^Ty+\mu(v_k+d_k)$.
4) $x_{k+1}=\frac{1}{\mu}(\alpha r_k+(1-\alpha)W^TWr_k-W^T\mathcal{F}Wr_k)$.
5) $v_{k+1}=\text{soft}(x_{k+1}-d_k,\frac{\lambda}{\mu})$.
6) $d_{k+1}=d_k-(x_{k+1}-v_{k+1})$.
7) $k\leftarrow k+1$.
8) *until stopping criterion is satisfied.*

Here $\alpha=\frac{\mu}{\mu+\gamma}$ and

$$\mathcal{F} \;=\; B^T(\mu I+BB^T)^{-1}B. \tag{6}$$

In the above algorithm, the multiplication by $\mathcal{F}$ can be regarded as a filtering in frame-domain. In general, its computation is not affordable if the matrix B has considerable large-size, thus for computing of x_{k+1}, we may take gradient-based algorithms which lead to the linear system being solved inexactly. However, for underlying MR image reconstruction, the matrix $B=SF$ has a special undersampled Fourier structure, where $S\in\mathbb{R}^{m\times n}$ is undersampling pattern, which can be formed by taking a subset of rows of an identity matrix. Hence we have $SS^T=I$. Using these facts and the Sherman-Morrison-Woodbury matrix inversion formula, we can obtain the following formula:

$$\mathcal{F}=F^TS^T(\mu I+SFF^TS^T)^{-1}SF=\frac{1}{1+\mu}F^TS^TSF. \tag{7}$$

Since S^TS is equal to an identity matrix with some zeros in the diagonal, it is a binary mask and can be computed with $O(n)$ cost. Then the products by F^T and F can be computed with $O(n\log n)$ cost using the fast Fourier transform. Furthermore, for a redundant tight wavelet frame, any matrix-vector multiplications can be performed by the fast $O(n\log n)$ implementations [6]. In conclusion, x_{k+1} can be computed with $O(n\log n)$ cost using the fast wavelet transform W and fast Fourier transform F.

In view of (7), we can obtain the following fast algorithm to solve the balanced regularization problem (4) for MR image reconstruction from undersampled measurements.

Algorithm ADMM for balanced regularization in MR image reconstruction **(MRI-ADMM-B)**:

1) Set $k=0$, choose $\mu>0$, v_0 and d_0.
2) *repeat*
3) $r_k=W^TB^Ty+\mu(v_k+d_k)$.
4) $x_{k+1}=\frac{1}{\mu}(\alpha r_k+(1-\alpha)W^TWr_k-\frac{1}{1+\mu}W^TF^TS^TSFWr_k)$.
5) $v_{k+1}=\text{soft}(x_{k+1}-d_k,\frac{\lambda}{\mu})$.
6) $d_{k+1}=d_k-(x_{k+1}-v_{k+1})$.
7) $k\leftarrow k+1$.
8) *until stopping criterion is satisfied.*

Let $\gamma=0$, we have the following algorithm to solve the synthesis-based approach in MR image reconstruction.

Algorithm ADMM for synthesis-based regularization in MR image reconstruction **(MRI-ADMM-S)**:

1) Set $k=0$, choose $\mu>0$, v_0 and d_0.
2) *repeat*
3) $r_k=W^TB^Ty+\mu(v_k+d_k)$.
4) $x_{k+1}=\frac{1}{\mu}(r_k-\frac{1}{1+\mu}W^TF^TS^TSFWr_k)$.
5) $v_{k+1}=\text{soft}(x_{k+1}-d_k,\frac{\lambda}{\mu})$.
6) $d_{k+1}=d_k-(x_{k+1}-v_{k+1})$.
7) $k\leftarrow k+1$.
8) *until stopping criterion is satisfied.*

Similarly, we can obtain the following algorithm for solving the analysis-based approach in MR image reconstruction ($\gamma=\infty$ in (4)).

Algorithm ADMM for analysis-based approach in MR image reconstruction **(MRI-ADMM-A)**:

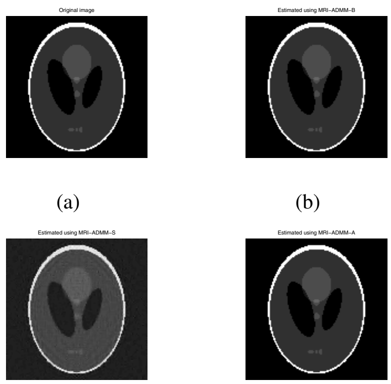
1) Set $k=0$, choose $\mu>0$, v_0 and d_0.
2) *repeat*
3) $r_k=B^Ty+\mu W(v_k+d_k)$.
4) $u_{k+1}=\frac{1}{\mu}(r_k-\frac{1}{1+\mu}F^TS^TSFr_k)$.
5) $v_{k+1}=\text{soft}(W^Tu_{k+1}-d_k,\frac{\lambda}{\mu})$.
6) $d_{k+1}=d_k-(W^Tu_{k+1}-v_{k+1})$.
7) $k\leftarrow k+1$.
8) *until stopping criterion is satisfied.*

It is noted that W^TB^Ty and B^Ty in the above algorithms do not change during iterations and can be precomputed. In the next section, the balanced approach is illustrated by experiments and compared with the synthesis-based and analysis-based approaches.

IV. Experimental Results

In this section, numerical results are reported to compare the performance of the balanced approach (MRI-ADMM-B) with the analysis-based (MRI-ADMM-A) and synthesis-based (MRI-ADMM-S) approaches for MR image reconstruction from undersampled measurements. Our simulations are written in MATLAB and performed on a Dell computer with Intel Xeon CPU 2.66GHz and 4GB of RAM and Windows XP. Moreover, in our experiments, the sparsifying frame W is a redundant Haar wavelet with four levels.

In this section, due to the space limitation, we only consider MR image reconstruction on the well-known Shepp-Logan phantom with size 128×128 shown in Fig. 1(a) from undersampled k-space data. The undersampled k-space data is generated from 44 radial lines of its 2-D discrete Fourier transform (sampling ratio 31.01%). The projections are also corrupted by circular complex Gaussian noise with variance $\sigma^2 = 0.5 \times 10^{-6}$. In our simulations, we empirically choose $\lambda = 10^{-4}$, $\gamma = 1$ and $\mu = 0.001$.



(a) (b)

(c) (d)

Fig. 1. (a) Original Shepp-Logan phantom. (b) Reconstructed image using balanced approach (MRI-ADMM-B). (c) Reconstructed image using Synthesis-based approach (MRI-ADMM-S). (d) Reconstructed image using Analysis-based approach (MRI-ADMM-A).

Figs. 1(b)–1(d) plot the reconstructed images by using balanced approach, synthesis-based and analysis-based approaches, respectively. From these figures, one can see that there were many noise residuals and artifacts around edges in the reconstructed Shepp-Logan phantom image by the synthesis-based approach. Both balanced approach and analysis-based approach can lead to much better visual quality.

To visually compare performance of the three approches, Fig. 2 gives the evolution of MSE ($= \frac{1}{N}\|u_k - u\|^2$ with u being the true image and N being its size). We can conclude from these figures that in MR image reconstruction, there exists an obvious gap between the synthesis-based and analysis-based approaches, and the balanced approach can reduce the gap and even produce better reconstructed MR image than these two approaches under our experimental conditions. It should be noted that our experimental results do not draw definitive conclusion as to which approach is better because each has its own favorable data sets and applications. But since the balanced approach can bridge the synthesis-based and analysis-based approaches and balance the fidelity, sparsity and smoothness of the solution, hence it is a good choice to use the balanced approach for frame-based image reconstruction when we do not know which approach is better.

V. Conclusions

Frame-based MR image reconstruction from undersampled k-space data has been presented by using a balanced approach, which bridge the synthesis-based and analysis-based approaches. These ℓ_1 regularization approaches for frame-based MR image reconstruction are solved by using a variable splitting and classical ADMM technique. Experimental results have shown that for the wavelet frame-based MR image reconstruction from undersampled measurements, the balanced approach reduces greatly the gap between the analysis-based and synthesis-based approaches and can provide much better quality image in our experimental cases. Although it is very hard to get a definitive conclusion which approach is better under different data sets and applications, the balanced approach is a good candidate when we do not know which approach is better for the frame-based MR image reconstruction.

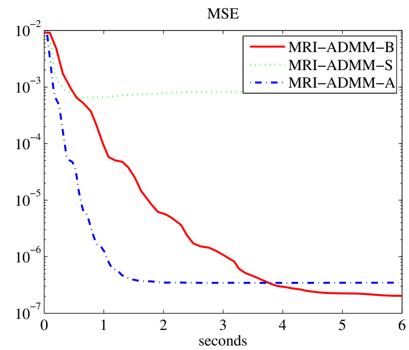


Fig. 2. Evolution of MSE over time.

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