Unconstrained Tracking MPC for Continuous-Time Nonlinear Systems

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Abstract

In this paper, we extend unconstrained model predictive control (MPC) from setpoint stabilization to dynamic reference tracking for continuous-time nonlinear systems. In particular, we focus on the case when the reference cannot be perfectly tracked by the system due to dynamics and/or constraints. Under the incremental stabilizability assumption and an additional dissipativity assumption, the practical stability of tracking the unknown optimal reachable reference trajectory is proved even though the controller does not know such a reference explicitly.

Key words: model predictive control; tracking control; nonlinear system; optimization.

1 Introduction

Model Predictive Control (MPC) has become one of the most popular control technologies in industry due to its capability to explicitly optimize performance index while satisfying state and input constraints. It solves a sequence of finite horizon optimal control problems and is implemented in a receding horizon manner to approximate an infinite horizon optimization, which is usually intractable.

In the past decades, setpoint stabilization of MPC has been studied extensively. The stability is ensured by properly designing terminal conditions (terminal sets and terminal costs) or adopting a sufficiently long prediction horizon without terminal conditions. For the first type, we refer [1] for the discrete-time case and [2] for the continuous-time case. For the second type, results on discrete-time systems can be found in [3], [4] and continuous-time cases are studied in [5].

A natural generalization of setpoint stabilization is reference tracking, which aims to drive the state or output of a system to follow a desired dynamic trajectory, and its application can be found in batch processes [6], mobile robots [7] and so on. Consequently, in recent years, reference tracking MPC were also studied. In [8], two robust MPC schemes have been designed for unicycle robots subject to bounded disturbances to track a (virtual) leader robot's trajectory, which is assumed to be reachable. The first tube based approach is an extension of the one proposed in [9]. It combines the open loop optimal control input with a linear feedback law based on the deviation of the actual state from the nominal one to force the state to evolve in a tube around the predicted trajectory. The second MPC extends the result in [10], which uses the robustness constraint to force the tracking error to decay at certain rate. A more complicated situation in reference tracking is that the desired trajectory may not be reachable by the system due to constraints and/or dynamics. One direct approach to overcome this issue is to calculate an optimal reachable trajectory offline, then the system aims to track the reachable one instead of the original unreachable one. A more interesting way is to integrate the offline path planning into the online control phase, i.e., the controller can drive the state or output of the system to the optimal reachable trajectory without computing it offline. In [11], the reference could be an arbitrary periodic trajectory and a single layer MPC unifying dynamic trajec-

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tory planning and tracking is proposed for discrete-time linear systems. Another single layer MPC is proposed in [12] for a discrete-time nonlinear system to track an arbitrary piece-wise constant reference. Both works share the same methodology, which introduces a virtual reference being optimized online and drives the system state and/or output to the virtual one.

All of the aforementioned works rely on properly designed terminal sets and terminal costs around the (virtual) reference signal. In [13], MPC without terminal conditions for setpoint stabilization has been extend to reference tracking of discrete-time nonlinear systems. For reachable cases, a lower bound of the prediction horizon is derived to ensure that the tracking error goes to zero. For unreachable cases, techniques from economic MPC (EMPC) is used to ensure practical stability of tracking the optimal reachable reference.

To the best of our knowledge, MPC without terminal conditions for reference tracking of continuous-time systems has not been studied yet. Note that many applications involve continuous-time models. In this paper, we extend the technique of EMPC without terminal conditions used in [14] and [13] to continuous-time systems and show the practical stability of the tracking error. In particular, we show that how the prediction horizon is related to the sampling time interval and provide a theoretical lower bound of the prediction horizon which ensures the practical stability. Compared with existing tracking MPC using reference-dependent terminal sets and/or terminal costs, the proposed approach does not require complex offline design and is more flexible when reference changes online.

The rest of this paper is organized as follows: In Section 2 we introduce an MPC tracking scheme and local incremental stabilizablity condition. In Section 3, the case of unreachable reference is studied and the practical stability of tracking the unknown optimal reachable trajectory is proved. In Section 4, the results are illustrated by a few numerical examples. Finally, some conclusions are drawn in Section 5.

Some remarks on notations are introduced as follows. We use $\mathbb{R}$ to denote the set of real numbers. $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$ and $\mathbb{N}$ denote n-dimensional Euclidean space, $m \times n$-dimensional Euclidean space and the set of natural numbers, respectively. For a matrix $A \in \mathbb{R}^{m\times n}$, A^T denotes its transpose. For a vector $x \in \mathbb{R}^n$, $\|x\|$ and $\|x\|_Q$ denote its 2-norm and Q-norm, i.e., $\|x\|_Q^2 = x^T Q x$, where Q is a positive definite matrix. For a real symmetric matrix Q, its largest and smallest eigenvalues are denoted as $\lambda_{\max}(Q)$ and $\lambda_{\min}(Q)$, respectively. $\mathcal{K}$ denotes the set of functions $\alpha(\cdot) : [0, \infty) \to [0, \infty)$, which are continuous, strictly increasing and satisfying $\alpha(0) = 0$. By $\mathcal{K}_\infty$ we denote the set of functions $\alpha(\cdot)$ belonging to $\mathcal{K}$ and satisfying $\lim_{r\to\infty} \alpha(r) = \infty$. $\mathcal{L}$ denotes functions $\beta : [0, \infty) \to [0, \infty)$, which are continuous and decreasing with $\lim_{r\to\infty} \beta(r) = 0$.

2 Problem Formulation and Preliminaries

We consider the following nonlinear continuous-time system

$$\dot{x}(t) = f(x(t), u(t)),\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{X} \subset \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{U} \subset \mathbb{R}^m$ is the control input, $\mathbb{Z} \triangleq \mathbb{X} \times \mathbb{U}$ is compact.

Given a reference signal $(x_r(t), u_r(t))$, the tracking error is defined as

$$l(x(t), u(t), t) = \|x(t) - x_r(t)\|_Q^2 + \|u(t) - u_r(t)\|_R^2, \tag{2}$$

where $Q = Q^T \in \mathbb{R}^{n\times n}$ is positive definite and $R = R^T \in \mathbb{R}^{m\times m}$ is semi-positive definite. If the reference control input is available, R can be chosen as a positive definite matrix. R can also be set as 0 if u_r is not available or $\|u(t) - u_r(t)\|$ is not considered as part of the performance index.

When the given reference is reachable, the setpoint stabilization [5] can be extended by incorporating the local controllability condition. In this paper, we mainly focus on a more difficult case when (x_r, u_r) is not reachable due to the constraints and system dynamics, i.e., (x_r, u_r) cannot be perfectly tracked by the system. In this case, the control objective is to drive system (1) to a reachable trajectory that optimizes some cost while satisfying the constraint $(x(t), u(t)) \in \mathbb{Z},\ t \geq 0$.

We propose the following MPC scheme to achieve our goal. Given a sampling interval $\delta > 0$, denote the sampling time instant $t_k \triangleq k\delta,\ \forall k \in \mathbb{N}$. At each sampling time instant t_k, the following open-loop constrained optimal control problem is solved:

Problem 1

$$\min J_T(x(t_k), t_k, u(\cdot|t_k)) = \int_{t_k}^{t_k+T} l(x(s|t_k), u(s|t_k), s)ds \tag{3}$$

subject to

$$\begin{aligned} \dot{x}(t|t_k) &= f(x(t|t_k), u(t|t_k)), \\ (u(t|t_k), x(t|t_k)) &\in \mathbb{Z},\ t \in [t_k, t_k + T], \\ x(t_k|t_k) &= x(t_k), \end{aligned}$$

where $T > \delta$ is the prediction horizon.

Denote the optimal solution of the above problem as $x^*(t|t_k)$ and $u^*(t|t_k)$, $t \in [t_k, t_k + T]$. The optimal value function is defined as $V_T(x(t_k), t_k) \triangleq J_T(x(t_k), t_k, u^*(\cdot|t_k))$. Then the control law is given by $u_{\text{MPC}}(t) = u^*(t|t_k), t \in [t_k, t_k + \delta)$.

Compared with most existing tracking MPC, we do not use terminal cost function and terminal constraint in the proposed optimization problem. In what follows, we are going to derive a few sufficient conditions on the system dynamics, reference trajectory, prediction horizon T and sampling interval δ under which the control goal can be achieved.

Assumption 2.1 *We assume that for a given $T > 0$, the infimun of **Problem 1** is attained and the corresponding* *stage cost* $l(x^*(s|t_k), u^*(s|t_k), s)$ *is piecewise continuous.*

We introduce the following local incremental stabilizability, which is a continuous-time version of the one introduced in [13]. A more general definition of incremental stability for continuous-time systems can be found in [15].

Assumption 2.2 *There exist a continuous control law* $\kappa : \mathbb{X} \times \mathbb{Z} \to \mathbb{R}^m$, *an* ϵ*-Lyapunov function* $V_\epsilon : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_{\geq 0}$, *which is continuous in the first argument and satisfies* $V_\epsilon(z, z) = 0$ *for all* $z \in \mathbb{X}$, *and positive constants* $c_{\epsilon,l}$, $c_{\epsilon,u}$, ϵ_{loc}, k_{max}, ρ, *such that for all initial condition* $(x(0), z(0)) \in \mathbb{X} \times \mathbb{X}$ *with* $V_\epsilon(x(0), z(0)) \leq \epsilon_{loc}$ *the following properties hold:*

$$c_{\epsilon,l}\|x - z\|^2 \leq V_\epsilon(x, z) \leq c_{\epsilon,u}\|x - z\|^2, \quad (4)$$

$$\|\kappa(x, z, v) - v\| \leq k_{max}\|x - z\|, \quad (5)$$

$$V_\epsilon(x(t), z(t)) \leq e^{-\rho(t-s)}V_\epsilon(x(s), z(s)),\ 0 \leq s \leq t \quad (6)$$

$$(z, v) \in \mathbb{Z},$$

with

$\dot{x} = f(x, \kappa(x, z, v)),\ \dot{z} = f(z, v).$

Remark 2.1 *Assumption 2.2 means that* (z, v) *can be perfectly tracked by using controller* $\kappa(x, z, v)$ *when* x *is sufficiently close to* z. *More specifically, it requires that* $\kappa(x, z, v)$ *can cancel out* v *when* x *is sufficiently close to* z. *Sufficient conditions under which* v *can be canceled out are given as follows:*

Assume that system dynamics f *is twice differentiable and the first-order Taylor-approximation of* f *around any point* $r = (z, v) \in \mathbb{Z}$ *can be written as*

$$f(z + \Delta x, v + \Delta u) = f(z, v) + A_r\Delta x + B_r\Delta u + \phi_r(\Delta x, \Delta u),$$

where $A_r = \frac{\partial f}{\partial x}|_{(z,v)}$, $B_r = \frac{\partial f}{\partial u}|_{(z,v)}$ *and* $\|\phi_r(\Delta x, \Delta u)\| \leq M(\|\Delta x\|^2 + \|\Delta u\|^2)$.

If for any point $r = (z, v) \in \mathbb{Z}$, *there exist a matrix* $K_r \in \mathbb{R}^{m\times n}$, *a positive constant* α *and positive definite matrices* P_r, $Q_r \in \mathbb{R}^{n\times n}$ *continuous in* r *such that*

$$(A_r + B_rK_r + \alpha I)^T P_r + P_r(A_r + B_rK_r + \alpha I) + \dot{P}_r + Q_r = 0,$$

then Assumption 2.2 can be satisfied by choosing $u = v + K_r(x - z)$ *and* $V_\epsilon(x, z) = \|x - z\|^2_{P_r}$.

A similar result can be found in [2] for setpoint stabilization problems. The difference is that the conditions in [2] are time invariant while for tracking cases, the linearized dynamics A_r, B_r, *the local feedback gain* K_r *and so on are dependent on the reference trajectory.*

3 Practical Reference Tracking with Economic MPC

In order to formulate the problem properly, we focus on reference with period T_p, i.e., $(x_r(t), u_r(t)) = (x_r(t + T_p), u_r(t+T_p))$. We introduce the following optimization problem:

Problem 2

$$\min_{x(0),u(\cdot)} \int_0^{T_p} (\|x(t) - x_r(t)\|^2_Q + \|u(t) - u_r(t)\|^2_R)dt$$

subject to

$$\dot{x}(t) = f(x(t), u(t)),$$
$$x(0) = x(T_p),$$
$$(x(t), u(t)) \in \mathbb{Z},$$
$$t \in [0, T_p].$$

Denote the optimal state and control trajectory of the above problem as $(x_p(t), u_p(t))$, the optimal value as $V_{T_p,\min}$, $c_{x,\sup} = \sup_{t\in[0,T_p]} \|x_p(t) - x_r(t)\|$ and $c_{c,\sup} = \sup_{t\in[0,T_p]} \|u_p(t) - u_r(t)\|$. Note that if the given reference trajectory $(x_r(t), u_r(t))$ is reachable, $(x_p(t), u_p(t)) = (x_r(t), u_r(t))$, i.e., $(x_r(t), u_r(t))$ can be perfectly tracked. Otherwise, $(x_p(t), u_p(t))$ is a trajectory different from $(x_r(t), u_r(t))$ but can be tracked by the system.

Now we make a stabilizability assumption with respect to $(x_p(t), u_p(t))$.

Assumption 3.1 *The optimal reachable reference trajectory* (x_p, u_p) *is such that* $V_\epsilon(x, x_p) \leq \epsilon_{p,ref}$ *implies*

$x \in \mathbb{X},\ \kappa(x, x_p, u_p) \in \mathbb{U}$

with V_ϵ *and* κ *from Assumption 2.2 and* $\epsilon_{p,ref} > 0$. *There exists a function* $\alpha_V \in \mathcal{K}$ *such that* $V_\epsilon(x, x_p) \leq \epsilon_{p,ref}$ *and* $V_\epsilon(y, x_p) \leq \epsilon_{p,ref}$ *implies that*

$|V_\epsilon(x, x_p) - V_\epsilon(y, x_p)| \leq \alpha_V(\|x - x_p\| + \|y - x_p\|).$

Denote $e_p = x - x_p$. We borrow the following dissipativity assumption from [16].

Assumption 3.2 *There exists a storage function* $\lambda : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *such that*

$$\lambda(t + \delta, x(t + \delta)) - \lambda(t, x(t)) \leq \int_t^{t+\delta} (s(\tau, x(\tau), u(\tau)) - \alpha_l(\|x(\tau) - x_p(\tau)\|))d\tau,$$

for all $t \geq 0, \delta > 0$, *with* $\alpha_l(\cdot)$ *being a class-*$\mathcal{K}_\infty$ *function and* $s(t, x(t), u(t)) = l(x(t), u(t), t) - l(x_p(t), u_p(t), t)$. *Furthermore,* $\lambda(t, x(t))$ *is uniformly bounded by*

$|\lambda(t, x(t))| \leq \gamma_\lambda(\|e_p(t)\|), \gamma_\lambda \in \mathcal{K}$.

Remark 3.1 *To explicitly construct a time-varying storage function* $\lambda(t, x)$ *for general nonlinear systems with respect to arbitrary reference trajectory* (x_r, u_r) *is difficult. Even for discrete-time setpoint stabilization, there is no systematic way to construct* λ *in general [17]. However, the existence of such a storage function can be shown by using local controllability assumption [18] and the definition of uniform suboptimal operation in [17]. The proof of Theorem 4 in [17] can be extended to the continuous-time tracking case directly and it is omitted here for conciseness.*

We define the rotated MPC problem as follows:

$$\min \tilde{J}_T(x(t_k), t_k, u(\cdot|t_k)) = \int_{t_k}^{t_k+T} (l(x(s|t_k), u(s|t_k), s) - l(x_p(s), u_p(s), s))ds - (\lambda(t_k + T, x(t_k + T|t_k)) - \lambda(t_k, x(t_k))) \quad (7)$$

subject to

$$\dot{x}(t|t_k) = f(x(t|t_k), u(t|t_k)),$$

$$(u(t|t_k),x(t|t_k)){\in}\mathbb{Z},\ t\in[t_k,t_k+T],$$
$$x(t_k|t_k){=}x(t_k),$$
$$t{\in}[t_k,t_k+T],$$

Denote the optimal state and control trajectory as $(\tilde{x}^*(s|t_k),\tilde{u}^*(s|t_k)),\ s\in[t_k,t_k+T]$ and the corresponding optimal value function as $\tilde{V}_T(x(t_k),t_k)$. We assume that $\tilde{V}_\infty(x(t),t)<\infty$ holds for all $x(t)\in\mathbb{X}$ and $t\geq 0$.

Since for a given prediction horizon T and time instant t_k, $\int_{t_k}^{t_k+T} l(x_p(s),u_p(s),s)ds$ is a constant, we denote $c_T(t_k)=\int_{t_k}^{t_k+T} l(x_p(s),u_p(s),s)ds$. Consequently, we have

$$\tilde{J}_T(x(t_k),t_k,u(\cdot|t_k)){=}J_T(x(t_k),t_k,u(\cdot|t_k))-c_T(t_k)$$
$$+\lambda(t_k,x(t_k)){-}\lambda(t_k{+}T,x(t_k{+}T|t_k)).$$

Proposition 3.1 *Let Assumptions 2.2, 3.1, and 3.2 be satisfied. Then there exist positive constants c_p, γ, $\tilde{c}_{max}$ and a function $\alpha_u\in\mathcal{K}$ such that for all $\|e_p(t_k)\|\leq c_p$, and all $T>0$, the following bounds hold*

$$V_T(x(t_k),t_k){\leq}\gamma\lambda_{max}(Q)\|e_p(t_k)\|^2+c_T(t_k)+\tilde{c}_{max}\|e_p(t_k)\|$$
$$\tilde{V}_T(x(t_k),t_k){\leq}\alpha_u(\|e_p(t_k)\|).$$

PROOF. Let $\epsilon_p=\min\{\epsilon_{\text{loc}},\epsilon_{p,\text{ref}}\}$ and $c_p=\sqrt{\frac{\delta_p}{c_{\epsilon,u}}}$. Then we have

$$V_\epsilon(x(t_k),x_p(t_k))\leq c_{\epsilon,u}\|e_p(t_k)\|^2\leq\epsilon_p.$$

Consider the candidate control and state trajectory given by

$$\bar{u}(t|t_k){=}\kappa(\bar{x}(t|t_k),x_p(t),u_p(t)),$$
$$\dot{\bar{x}}(t|t_k){=}f(\bar{x}(t|t_k),\bar{u}(t|t_k)),$$
$$\bar{x}(t_k|t_k){=}x(t_k),t\geq t_k,$$
$$\bar{e}_p(t|t_k){=}\bar{x}(t|t_k)-x_p(t).$$

By (6), we have

$$V_\epsilon(\bar{x}(t|t_k),x_p(t)){\leq}e^{-\rho(t-t_k)}V_\epsilon(\bar{x}(t_k),x_p(t_k))\leq\epsilon_{p,\text{ref}}.$$

According to Assumption 3.1, we have $\bar{x}(t|t_k)\in\mathbb{X}$ and $\bar{u}(t|t_k)\in\mathbb{U}$. Therefore, the candidate trajectory is feasible. The stage cost $l(\bar{x}(s|t_k),\bar{u}(s|t_k),s)$ can be bounded as

$$\begin{aligned}
&l(\bar{x}(s|t_k),\bar{u}(s|t_k),s)\\
=&\|\bar{x}(s|t_k)-x_r(s)\|_Q^2+\|\bar{u}(s|t_k)-u_r(s)\|_R^2\\
\leq&\|\bar{x}(s|t_k){-}x_p(s)\|_Q^2{+}\|u_p(s){-}u_r(s)\|_R^2{+}\|x_p(s){-}x_r(s)\|_Q^2\\
&+2\lambda_{\max}(Q)\|\bar{x}(s|t_k)-x_p(s)\|\|x_p(s)-x_r(s)\|\\
&+k_{\max}^2\lambda_{\max}(R)\|\bar{x}(s|t_k)-x_p(s)\|^2\\
&+2\lambda_{\max}(R)\|\bar{u}(s|t_k)-u_p(s)\|\|u_p(s)-u_r(s)\|\\
\leq&(1+\frac{k_{\max}^2\lambda_{\max}(R)}{\lambda_{\min}(Q)})\|\bar{e}(s|t_k)\|_Q^2\\
&+2(\lambda_{\max}(Q)c_{x,\sup}+\lambda_{\max}(R)c_{u,\sup}k_{\max})\|\bar{e}(s|t_k)\|\\
&+\|x_p(s)-x_r(s)\|_Q^2+\|u_p(s)-u_r(s)\|_R^2.
\end{aligned}$$

By some simple calculations, we have

$$\int_{t_k}^{t_k+T}(1+\frac{k_{\max}^2\lambda_{\max}(R)}{\lambda_{\min}(Q)})\|\bar{e}(s|t_k)\|_Q^2ds$$
$$\leq\frac{C}{\rho}\|e_p(t_k)\|_Q^2\leq\gamma\|e_p(t_k)\|^2,$$

where $C=\frac{\lambda_{\max}(Q)c_{\epsilon,u}}{\lambda_{\min}(Q)c_{\epsilon,l}}(1+\frac{\lambda_{\max}(R)k_{\max}^2}{\lambda_{\min}(Q)})$ and $\gamma=\frac{C}{\rho}$.

(4) and (6) imply that

$$\sqrt{c_{\epsilon,l}}\|\bar{x}(s|t_k)-x_p(s)\|{\leq}V_\epsilon^{1/2}(\bar{x}(s|t_k),x_p(s))$$
$${\leq}\sqrt{c_{\epsilon,u}}\|\bar{x}(s|t_k)-x_p(s)\|,$$

and

$$\frac{dV_\epsilon^{1/2}(\bar{x}(s|t_k),x_p(s))}{ds}\leq-\frac{\rho}{2}V_\epsilon^{1/2}(\bar{x}(s|t_k),x_p(s)),$$

which results in

$$\|\bar{e}(s|t_k)\|\leq\sqrt{\frac{c_{\epsilon,u}}{c_{\epsilon,l}}}e^{-\frac{1}{2}\rho(s-t_k)}\|e_p(t_k)\|,$$

leading to

$$\int_{t_k}^{t_k+T}2(\lambda_{\max}(Q)c_{x,\sup}{+}\lambda_{\max}(R)c_{u,\sup}k_{\max})\|\bar{e}(s|t_k)\|ds$$
$$\leq\tilde{c}_{\max}\|e_p(t_k)\|,$$

where $\tilde{c}_{\max}=\frac{4}{\rho}(\lambda_{\max}(Q)c_{x,\sup}{+}\lambda_{\max}(R)c_{u,\sup}k_{\max})\sqrt{\frac{c_{\epsilon,u}}{c_{\epsilon,l}}}$.

Finally, note that

$$\int_{t_k}^{t_k+T}\|x_p(s)-x_r(s)\|_Q^2+\|u_p(s)-u_r(s)\|_R^2ds=c_T(t_k).$$

The first inequality is proved.

For the rotated cost, we have

$$\begin{aligned}
&\tilde{J}(x(t_k),t_k,\bar{u}(\cdot|t_k))\\
\leq&\gamma\|e_p(t_k)\|_Q^2+\tilde{c}_{\max}\|e_p(t_k)\|+\lambda(t_k,x(t_k))\\
&-\lambda(t_k+T,\bar{x}(t_k+T|t_k))\\
\leq&\gamma\|e_p(t_k)\|_Q^2+\tilde{c}_{\max}\|e_p(t_k)\|+\gamma_\lambda(\|e_p(t_k)\|)\\
&+\gamma_\lambda(\|e_p(t_k+T|t_k)\|)\\
\leq&\gamma\|e_p(t_k)\|_Q^2+\tilde{c}_{\max}\|e_p(t_k)\|+\gamma_\lambda(\|e_p(t_k)\|)\\
&+\gamma_\lambda(\sqrt{\frac{c_{\epsilon,u}}{c_{\epsilon,l}}}e^{-\frac{1}{2}\rho T}\|e_p(t_k)\|)\\
\leq&\gamma\lambda_{\max}(Q)\|e_p(t_k)\|^2+\tilde{c}_{\max}\|e_p(t_k)\|+\gamma_\lambda(\|e_p(t_k)\|)\\
&+\gamma_\lambda(\sqrt{\frac{c_{\epsilon,u}}{c_{\epsilon,l}}}\|e_p(t_k)\|)\\
:=&\alpha_u(\|e_p(t_k)\|).
\end{aligned}$$

Lemma 3.1 *Let Assumption 2.2, 3.1 and 3.2 hold. There exist functions σ, $\tilde{\sigma}\in\mathcal{L}$, such that the following turnpike property holds for all positive $\tilde{T}$, T with $\tilde{T}\leq T$ and all $\|e_p(t_k)\|\leq c_p$ with c_p defined in Proposition 3.1:*
1) There exist time intervals over $[t_k,t_k+T]$ with total length of at least $\tilde{T}$ and over which

$$\|e_p^*(s|t_k)\|\leq\sigma(T-\tilde{T}).$$

2) There exist time intervals over $[t_k,t_k+T]$ with total length of at least T' and over which

$$\|e_p^*(s|t_k)\|\leq\tilde{\sigma}(T-T'),\|\tilde{e}_p^*(s|t_k)\|\leq\tilde{\sigma}(T-T')$$

hold simultaneously. The corresponding rotated open-loop costs from t_k to s satisfy

$$\tilde{J}_{s-t_k}(x(t_k),t_k,u^*(\cdot|t_k))-\tilde{J}_{s-t_k}(x(t_k),t_k,\tilde{u}^*(\cdot|t_k))$$

$$\leq 2\gamma_\lambda(\tilde{\sigma}(T-T')) + \alpha_V(2\tilde{\sigma}(T-T')).$$

PROOF. 1) We first bound the rotated cost of the optimal solution to (3) as follows:

$$\begin{aligned}&\tilde{J}_T(x(t_k), t_k, u^*(\cdot|t_k))\\ =&V_T(x(t_k), t_k) - c_T(t_k) + \lambda(t_k, x(t_k))\\ &-\lambda(t_k+T, x^*(t_k+T|t_k))\\ \leq&\gamma\lambda_{\max}(Q)\|e_p(t_k)\|^2 + \tilde{c}_{\max}\|e_p(t_k)\|\\ &+\gamma_\lambda(\|e_p(t_k)\|) + \gamma_\lambda(\|e_p^*(t_k+T|t_k)\|)\\ \leq&\alpha_u(\|e_p(t_k)\|) + C \leq \alpha_u(c_p) + C, \end{aligned} \tag{8}$$

where $C = \sup_{x_1, x_2 \in \mathbb{X}} \gamma_\lambda(\|x_1 - x_2\|)$. Therefore, by the optimality of $\tilde{V}_T(x(t_k), t_k)$, we have

$$\begin{aligned}&\tilde{V}_T(x(t_k), t_k) \leq \tilde{J}_T(x(t_k), t_k, u^*(\cdot|t_k))\\ \leq&\alpha_u(\|e_p(t_k)\|) + C \leq \alpha_u(c_p) + C.\end{aligned}$$

Define

$$\sigma(T-\tilde{T}) = \alpha_l^{-1}(\frac{\alpha_u(c_p)+C}{T-\tilde{T}}).$$

Suppose that the total length of the time intervals over which $\|e_p^*(s|t_k)\| > \sigma(T-\tilde{T})$ is longer than $T-\tilde{T}$. By Assumption 3.2, we know that for any interval $[a,b] \subset [t_k, t_k+T]$, we have

$$\begin{aligned}&\int_a^b (l(x(s|t_k), u(s|t_k), s) - l(x_p(s), u_p(s), s))ds\\ &-(\lambda(b, x(b|t_k)) - \lambda(a, x(a)))\\ \geq&\int_a^b \alpha_l(\|x(s|t_k) - x_p(s)\|)ds \geq 0.\end{aligned}$$

Therefore, if we denote the union of all the time intervals over which $\|e_p^*(s|t_k)\| > \sigma(T-\tilde{T})$ as $\mathcal{T}$, we can write that

$$\begin{aligned}&\tilde{J}_T(x(t_k), t_k, u^*(\cdot|t_k)) \geq \int_{\mathcal{T}} \alpha_l(\|x^*(s|t_k) - x_p(s)\|)ds\\ &> \alpha_u(c_p) + C,\end{aligned}$$

which contradicts (8). Thus, the total length of the time intervals over which $\|e_p^*(s|t_k)\| \leq \sigma(T-\tilde{T})$ is at least $\tilde{T}$.

2) Similarly, the total length of the time intervals over which $\|\tilde{e}_p^*(s|t_k)\| \leq \sigma(T-\tilde{T})$ is also at least $\tilde{T}$. So, for any given $T_0 < \frac{1}{2}T$, $\|e_p^*(s|t_k)\| > \sigma(T_0)$ for time intervals with total length at most T_0 and $\|\tilde{e}_p^*(s|t_k)\| > \sigma(T_0)$ for time intervals with total length at most T_0. Then, the total length of the time intervals over which

$$\|e_p^*(s|t_k)\| \leq \tilde{\sigma}(T-T'), \ \|\tilde{e}_p^*(s|t_k)\| \leq \tilde{\sigma}(T-T')$$

hold simultaneously is at least $T' = T - 2T_0$, where

$$\tilde{\sigma}(T-T') = \alpha_l^{-1}(2\frac{\alpha_u(c_p)+C}{T-T'}).$$

Denote the set of time instants over which $\|e_p^*(s|t_k)\| \leq \tilde{\sigma}(T-T')$ and $\|\tilde{e}_p^*(s|t_k)\| \leq \tilde{\sigma}(T-T')$ hold simultaneously as $\mathcal{T}'$ and pick arbitrary time instant s in $\mathcal{T}'$. We have

$$\begin{aligned}&\tilde{J}_{s-t_k}(x(t_k), t_k, u^*(\cdot|t_k))\\ =&V_T(x(t_k), t_k) + \lambda(t_k, x(t_k)) - \lambda(s, x^*(s|t_k))\\ &-c_{s-t_k}(t_k) - V_{T-s+t_k}(x^*(s|t_k), s)\\ \leq&J_{s-t_k}(x(t_k), t_k, \tilde{u}^*(\cdot|t_k)) + \lambda(t_k, x(t_k)) - \lambda(s, x^*(s|t_k))\\ &-c_{s-t_k}(t_k) + V_{T-s+t_k}(\tilde{x}^*(s|t_k), s) - V_{T-s+t_k}(x^*(s|t_k), s)\\ =&\tilde{J}_{s-t_k}(x(t_k), t_k, \tilde{u}^*(\cdot|t_k)) - \lambda(s, x^*(s|t_k)) + \lambda(s, \tilde{x}^*(s|t_k))\\ &+V_{T-s+t_k}(\tilde{x}^*(s|t_k), s) - V_{T-s+t_k}(x^*(s|t_k), s)\\ \leq&\tilde{J}_{s-t_k}(x(t_k), t_k, \tilde{u}^*(\cdot|t_k)) + 2\gamma_\lambda(\tilde{\sigma}(T-T'))\\ &+\alpha_V(2\tilde{\sigma}(T-T')).\end{aligned}$$

Lemma 3.2 *Let Assumption 3.2 hold. For any $\tilde{V}_{max} > 0$, there exists a function $\tilde{\sigma}_{\tilde{V}_{max}} \in \mathcal{L}$ such that for any $x(t_k)$ with $\tilde{V}_T(x(t_k), t_k) \leq \tilde{V}_{max}$ and any positive T', T with $T' < T$, the total length of time intervals over which $\|e_p^*(s|t_k)\| \leq \tilde{\sigma}_{\tilde{V}_{max}}(T-T')$, $\|\tilde{e}_p^*(s|t_k)\| \leq \tilde{\sigma}_{\tilde{V}_{max}}(T-T')$ hold simultaneously is at least T'. The corresponding open-loop costs satisfy*

$$\begin{aligned}&\tilde{J}_{s-t_k}(x(t_k), t_k, u^*(\cdot|t_k)) - \tilde{J}_{s-t_k}(x(t_k), t_k, \tilde{u}^*(\cdot|t_k))\\ \leq&2\gamma_\lambda(\tilde{\sigma}_{\tilde{V}_{max}}(T-T')) + \alpha_V(2\tilde{\sigma}_{\tilde{V}_{max}}(T-T')).\end{aligned}$$

PROOF. For the rotated optimal value function $\tilde{V}_T(x(t_k), t_k)$ we have

$$\begin{aligned}\tilde{V}_T(x(t_k), t_k)=&J_T(x(t_k), t_k, \tilde{u}^*(\cdot|t_k)) - c_T(t_k)\\ &+\lambda(t_k, x(t_k)) - \lambda(t_k+T, \tilde{x}^*(t_k+T|t_k)).\end{aligned}$$

Then $\tilde{J}_T(x(t_k), t_k, u^*(\cdot|t_k))$ can be bounded as follows:

$$\begin{aligned}&\tilde{J}_T(x(t_k), t_k, u^*(\cdot|t_k))\\ =&V_T(x(t_k), t_k) - c_T(t_k) + \lambda(t_k, x(t_k))\\ &-\lambda(t_k+T, x^*(t_k+T|t_k))\\ \leq&J_T(x(t_k), t_k, \tilde{u}^*(\cdot|t_k)) - c_T(t_k) + \lambda(t_k, x(t_k))\\ &-\lambda(t_k+T, x^*(t_k+T|t_k))\\ =&\tilde{V}_T(x(t_k), t_k) + \lambda(t_k+T, \tilde{x}^*(t_k+T|t_k))\\ &-\lambda(t_k+T, x^*(t_k+T|t_k))\\ \leq&\tilde{V}_{\max} + 2C,\end{aligned}$$

where C is defined in (8). The rest of the proof follows the same line of the proof of Lemma 3.1 with

$$\tilde{\sigma}_{\tilde{V}_{\max}}(T-T') = \alpha_l^{-1}(2\frac{\tilde{V}_{\max}+2C}{T-T'}).$$

Assumption 3.3 *For any given positive constant $\delta \leq T$, there exists a function $\alpha_\delta \in \mathcal{K}$ satisfying that*

$$\int_{t_k}^{t_k+\delta} \alpha_l(\|x(s|t_k) - x_p(s)\|)ds \geq \alpha_\delta(\|e_p(t_k)\|),$$

for all feasible $x(s|t_k)$, $s \in [t_k, t_k+T]$, where $\alpha_{\delta_1}(r) \geq \alpha_{\delta_2}(r)$, if $\delta_1 \geq \delta_2$.

Remark 3.2 *Assumption 3.3 requires that the optimal cost be lower bounded by a $\mathcal{K}$ function of the initial error state. Construction of such a lower bound for polynomial systems using convex optimization can be found in [19] and for piecewise linear systems can be found in [20].*

We introduce the set $\mathbb{S}_{c_p}^\delta = \{(x,t)|\tilde{V}_T(x,t) \leq \alpha_\delta(c_p)\}$.

Theorem 3.1 *Let Assumptions 2.2, 3.1, 3.2 and 3.3 hold. Then there exist $\tilde{T}_0$ and a function $\tilde{\theta} \in \mathcal{L}$, such that for all $T > \tilde{T}_0$ and all initial conditions satisfying*

$(x(0),0) \in \mathbb{S}_{c_p}^{\delta}$, ***Problem 1*** *is recursively feasible and the closed-loop system satisfies*

$$\alpha_\delta(\|e_p(t_k)\|) \le \tilde{V}_T(x(t_k),t_k) \le \alpha_u(\|e_p(t_k)\|), \quad (9)$$

$$\tilde{V}_T(x(t_{k+1}),t_{k+1}) - \tilde{V}_T(x(t_k),t_k) \le -\alpha_\delta(\|e_p(t_k)\|) + \tilde{\theta}(T-\delta), \quad (10)$$

$$(x(t_k),t_k) \in \mathbb{S}_{c_p}^{\delta},$$

for all $k \in \mathbb{N}$.

PROOF. By Assumptions 3.2 and 3.3, we have

$$\tilde{V}_T(x(0),0) \ge \int_0^T \alpha_l(\|\tilde{x}^*(\tau|0) - x_p(\tau)\|)d\tau \ge \alpha_\delta(\|e_p(0)\|).$$

Since $(x(0),0) \in \mathbb{S}_{c_p}^{\delta}$, $\alpha_\delta(c_p) \ge \tilde{V}_T(x(0),0)$ leads to that $\|e_p(0)\| \le c_p$. Then the upper bound in (9) follows from Proposition 3.1.

Now we take $T' = \delta + \xi$ for arbitrarily small $\xi > 0$ in Lemma 3.1, which implies that there exists some time instant t^* over $(\delta, T]$ when

$$\|e_p^*(t^*|0)\| \le \tilde{\sigma}(T-T'),$$
$$\|\tilde{e}_p^*(t^*|0)\| \le \tilde{\sigma}(T-T'),$$

hold simultaneously. Therefore, for $T > T_1 \triangleq \tilde{\sigma}^{-1}(c_p) + T'$, there exists $t^* \in (\delta, T]$ such that $\|e_p^*(t^*|0)\| \le c_p$ and $\|\tilde{e}_p^*(t^*|0)\| \le c_p$. A feasible solution for the optimization problem formulated at time instant δ can be constructed as

$$\bar{u}(t|\delta) = \begin{cases} u^*(t|0), \ \delta \le t \le t^* \\ \kappa(\bar{x}(t|\delta), x_p(t), u_p(t)), \ t^* < t \le T+\delta \end{cases}$$

where $\dot{\bar{x}}(t|\delta) = f(\bar{x}(t|\delta), \bar{u}(t|\delta))$. By the principle of optimality

$$\begin{aligned}\tilde{V}_T(x(\delta),\delta) \le & \int_\delta^{t^*} (l(x^*(s|\delta),u^*(s|\delta),s) - l(x_p(s),u_p(s),s))ds \\ & -(\lambda(t^*, x^*(t^*|0)) - \lambda(\delta, x(\delta))) \\ & +\tilde{V}_{T+\delta-t^*}(x^*(t^*|0), t^*).\end{aligned}$$

Note that

$$\begin{aligned}&\int_\delta^{t^*} (l(x^*(s|\delta),u^*(s|\delta),s) - l(x_p(s),u_p(s),s))ds \\ &-(\lambda(t^*, x^*(t^*|0)) - \lambda(\delta, x(\delta))) \\ =& -\int_0^\delta (l(x^*(s|\delta),u^*(s|\delta),s) - l(x_p(s),u_p(s),s))ds \\ &+\lambda(\delta,x(\delta)) - \lambda(0,x(0)) + \tilde{J}_{t^*}(x(0),0,u^*(\cdot|0)),\end{aligned}$$

and

$$\begin{aligned}&\int_0^\delta (l(x^*(s|\delta),u^*(s|\delta),s) - l(x_p(s),u_p(s),s))ds \\ &-\lambda(\delta,x(\delta)) + \lambda(0,x(0)) \\ \ge& \int_0^\delta \alpha_l(\|e_p^*(s|0)\|)ds \ge \alpha_\delta(\|e_p(0)\|).\end{aligned}$$

Applying Lemma 3.1 leads to that

$$\begin{aligned}&\tilde{V}_T(x(\delta),\delta) \\ \le& \tilde{J}_{t^*}(x(0),0,u^*(\cdot|0)) + \lambda(\delta,x(\delta)) - \lambda(0,x(0)) \\ &-\int_0^\delta (l(x^*(s|\delta),u^*(s|\delta),s) - l(x_p(s),u_p(s),s))ds \\ &+\tilde{V}_{T+\delta-t^*}(x^*(t^*|0),t^*) \\ \le& \tilde{J}_{t^*}(x(0),0,\tilde{u}^*(\cdot|0)) + 2\gamma_\lambda(\tilde{\sigma}(T-T')) \\ &+\alpha_V(2\tilde{\sigma}(T-T')) - l(x_p(s),u_p(s),s))ds \\ &-\int_0^\delta (l(x^*(s|\delta),u^*(s|\delta),s) \\ &+\lambda(\delta,x(\delta)) - \lambda(0,x(0)) + \alpha_u(\|e_p^*(t^*|0)\|) \\ \le& \tilde{V}_T(x(0),0) + \lambda(\delta,x(\delta)) - \lambda(0,x(0)) + \tilde{\theta}(T-T') \\ &-\int_0^\delta (l(x^*(s|\delta),u^*(s|\delta),s) - l(x_p(s),u_p(s),s))ds, \quad (11) \\ \le& \tilde{V}_T(x(0),0) - \alpha_\delta(\|e_p(0)\|) + \tilde{\theta}(T-T'),\end{aligned}$$

where $\tilde{\theta}(T) = 2\gamma_\lambda(\tilde{\sigma}(T)) + \alpha_V(2\tilde{\sigma}(T)) + \alpha_u(\tilde{\sigma}(T))$. Note that $T' = \delta + \xi$. Therefore,

$$\tilde{V}_T(x(\delta),\delta) \le \tilde{V}_T(x(0),0) - \alpha_\delta(\|e_p(0)\|) + \tilde{\theta}(T-\delta-\xi)$$

holds for any $\xi > 0$. Then by the continuity of $\tilde{\theta}$, we have

$$\tilde{V}_T(x(\delta),\delta) \le \tilde{V}_T(x(0),0) - \alpha_\delta(\|e_p(0)\|) + \tilde{\theta}(T-\delta).$$

Finally, we need to ensure $\tilde{V}_T(x(\delta),\delta) \le \alpha_\delta(c_p)$ such that the proof can be concluded by induction. To this end, we consider the quantity $\alpha_\delta^{-1}(\tilde{\theta}(T-\delta))$ and the following two cases:

Case 1: $\|e_p(0)\| \ge \alpha_\delta^{-1}(\tilde{\theta}(T-\delta))$. In this case, we have

$$\begin{aligned}\tilde{V}_T(x(\delta),\delta) \le& \tilde{V}_T(x(0),0) - \alpha_\delta(\|e_p(0)\|) + \tilde{\theta}(T-\delta) \\ \le& \tilde{V}_T(x(0),0) - \tilde{\theta}(T-\delta) + \tilde{\theta}(T-\delta) \\ \le& \alpha_\delta(c_p).\end{aligned}$$

Case 2: $\|e_p(0)\| < \alpha_\delta^{-1}(\tilde{\theta}(T-\delta))$. In this case, we have

$$\begin{aligned}\tilde{V}_T(x(\delta),\delta) \le& \tilde{V}_T(x(0),0) - \alpha_\delta(\|e_p(0)\|) + \tilde{\theta}(T-\delta) \\ <& \alpha_u(\alpha_\delta^{-1}(\tilde{\theta}(T-\delta))) + \tilde{\theta}(T-\delta) \\ \triangleq& \tilde{\Theta}(T-\delta).\end{aligned}$$

Therefore, if $T > T_2 \triangleq \tilde{\Theta}^{-1}(\alpha_\delta(c_p)) + \delta$, then for both the cases we have $\tilde{V}_T(x(\delta),\delta) \le \alpha_\delta(c_p)$. In summary, we have shown that if **Problem 1** is feasible at $t_k = 0$ and $(x(0),0) \in \mathbb{S}_{c_p}^{\delta}$, then inequalities (9) and (10) are satisfied. Furthermore, **Problem 1** is feasible at the next time instant $t_k = \delta$ and $(x(\delta),\delta) \in \mathbb{S}_{c_p}^{\delta}$. Thus, the desired results can be derived by induction.

Theorem 3.1 ensures practical stability of tracking the unknown optimal reachable reference trajectory when the initial condition belongs to a possibly small region of contraction $\mathbb{S}_{c_p}^{\delta}$. The next theorem extends the practical stability result to a larger region of contraction.

Theorem 3.2 *Let Assumptions 2.2, 3.1, 3.2 and 3.3 hold. Then for any* $\tilde{V}_{max} \ge \alpha_\delta(c_p) > 0$, *there exist* $\tilde{T}_1$ *and a function* $\alpha_{u,\tilde{V}_{max}} \in \mathcal{K}$, *such that for all* $T > \tilde{T}_1$ *and all initial conditions satisfying* $\tilde{V}_T(x(0),0) \le \tilde{V}_{max}$, ***Problem 1*** *is recursively feasible and the closed-loop system* *satisfies*

$$\alpha_\delta(\|e_p(t_k)\|) \le \tilde{V}_T(x(t_k), t_k) \le \alpha_{u,\tilde{V}_{max}}(\|e_p(t_k)\|), \quad (12)$$

$$\tilde{V}_T(x(t_{k+1}), t_{k+1}) - \tilde{V}_T(x(t_k), t_k) \le -\alpha_\delta(\|e_p(t_k)\|) + \tilde{\theta}_{max}(T-\delta), \quad (13)$$

with $\tilde{\theta}_{max} \in \mathcal{L}$. *Furthermore,* $(x(t_k), t_k)$ *enters* $\mathbb{S}^\delta_{c_p}$ *within a finite number of steps and stays inside thereafter.*

PROOF. For any given $\beta > 0$, define

$$\alpha_{u,\tilde{V}_{\max}}(r) = \begin{cases} \max\{\alpha_u(r), \frac{r}{c_p}\tilde{V}_{\max}\}, & \text{if } r \le c_p \\ \max\{\alpha_u(c_p), \tilde{V}_{\max}\} + \beta(r - c_p), & \text{else.} \end{cases}$$

Then if $\|e_p(0)\| \le c_p$, by Lemma 3.1 $\alpha_{u,\tilde{V}_{\max}}(\|e_p(0)\|) \ge \alpha_u(\|e_p(0)\|) \ge \tilde{V}_T(x(0), 0)$. If $\|e_p(0)\| > c_p$, then $\alpha_{u,\tilde{V}_{\max}}(\|e_p(0)\|) \ge \tilde{V}_{\max} \ge \tilde{V}_T(x(0), 0)$. Therefore, we have $\tilde{V}_T(x(0), 0) \le \alpha_{u,\tilde{V}_{\max}}(\|e_p(0)\|)$. The lower bound holds due to Assumption 3.2 and 3.3.

Consider the quantity $\tilde{c}_p = \alpha_u^{-1}(\alpha_\delta(c_p)) \le c_p$ and the following two cases:

Case 1: $\|e_p(0)\| \le \tilde{c}_p \le c_p$. In this case we can apply Theorem 3.1 to have

$$\tilde{V}_T(x(\delta), \delta) - \tilde{V}_T(x(0), 0) \le -\alpha_\delta(\|e_p(0)\|) + \tilde{\theta}(T - \delta),$$

and $\tilde{V}_T(x(\delta), \delta) \le \tilde{V}_{\max}$ for all $T > \tilde{T}_0$.

Case 2: $\|e_p(0)\| > \tilde{c}_p$. We take $T' = \delta + \xi$ for arbitrarily small $\xi > 0$ in Lemma 3.2, which implies that there exists some time instant $t^* \in (\delta, T]$ such that

$$\|e_p^*(t^*|0)\| \le \tilde{\sigma}_{\tilde{V}_{\max}}(T - T'),\ \|\tilde{e}_p^*(t^*|0)\| \le \tilde{\sigma}_{\tilde{V}_{\max}}(T - T')$$

hold simultaneously. Therefore, for $T > T_3 \triangleq \tilde{\sigma}^{-1}_{\tilde{V}_{\max}}(\tilde{c}_p) + T'$ we have $\|e_p^*(t^*|0)\| < \tilde{c}_p$ and $\|\tilde{e}_p^*(t^*|0)\| < \tilde{c}_p$. Proposition 3.1 ensures that

$$\tilde{V}_{T+\delta-t^*}(x^*(t^*|0), t^*) \le \alpha_u(\|e_p^*(t^*|0)\|).$$

Similar to Theorem 3.1, a feasible solution for the optimization problem formulated at time instant δ can be constructed as

$$\bar{u}(t|\delta) = \begin{cases} u^*(t|0), & \delta \le t \le t^* \\ \kappa(\bar{x}(t|\delta), x_p(t), u_p(t)), & t^* < t \le T + \delta \end{cases}$$

where $\dot{\bar{x}}(t|\delta) = f(\bar{x}(t|\delta), \bar{u}(t|\delta))$. Using Lemma 3.2 we can further derive that

$$\begin{aligned} &\tilde{V}_T(x(\delta), \delta) \\ \le& \tilde{J}_{t^*}(x(0), 0, u^*(\cdot|0)) - \alpha_\delta(\|e_p(0)\|) + \tilde{V}_{T+\delta-t^*}(x^*(t^*|0), t^*) \\ \le& \tilde{J}_{t^*}(x(0), 0, \tilde{u}^*(\cdot|0)) + 2\gamma_\lambda(\tilde{\sigma}_{\tilde{V}_{\max}}(T - T')) \\ &+ \alpha_V(2\tilde{\sigma}_{\tilde{V}_{\max}}(T - T')) - \alpha_\delta(\|e_p(0)\|) + \alpha_u(\|e_p^*(t^*|0)\|) \\ \le& \tilde{V}_T(x(0), 0) - \alpha_\delta(\|e_p(0)\|) + \tilde{\theta}_{\tilde{V}_{\max}}(T - T'), \end{aligned}$$

where $\tilde{\theta}_{\tilde{V}_{\max}}(T) = 2\gamma_\lambda(\tilde{\sigma}_{\tilde{V}_{\max}}(T)) + \alpha_V(2\tilde{\sigma}_{\tilde{V}_{\max}}(T)) + \alpha_u(\tilde{\sigma}_{\tilde{V}_{\max}}(T)) = \tilde{\theta}(\tilde{\sigma}^{-1}(\tilde{\sigma}_{\tilde{V}_{\max}}(T)))$.

For $T > T_4 \triangleq \tilde{\theta}^{-1}_{\tilde{V}_{\max}}(\alpha_\delta(\tilde{c}_p)) + T'$, we have

$$\tilde{\theta}_{\tilde{V}_{\max}}(T - T') < \alpha_\delta(\tilde{c}_p) < \alpha_\delta(\|e_p(0)\|),$$

and

$$\tilde{V}_T(x(\delta), \delta) \le -\xi_{T,T'} + \tilde{V}_T(x(0), 0) < \tilde{V}_{\max}, \quad (14)$$

where $\xi_{T,T'} = \alpha_\delta(\tilde{c}_p) - \tilde{\theta}_{\tilde{V}_{\max}}(T - T') > 0$.

Combining Case 1 and 2 leads to that for all $T > \tilde{T}_1 \triangleq \max\{\tilde{T}_0, T_3, T_4\}$,

$$\tilde{V}_T(x(\delta), \delta) - \tilde{V}_T(x(0), 0) \le -\alpha_\delta(\|e_p(0)\|) + \tilde{\theta}_{\max}(T - T'),$$

and $\tilde{V}_T(x(\delta), \delta) \le \tilde{V}_{\max}$ with $\tilde{\theta}_{\max}(T - T') = \max\{\tilde{\theta}(T - T'), \tilde{\theta}_{\tilde{V}_{\max}}(T - T')\}$. In summary, we have shown that if **Problem 1** is feasible at $t_k = 0$ and $(x(0), 0) \in \{(x, t)|\tilde{V}_T(x(t), t) \le \tilde{V}_{\max}\}$, then inequalities (12) and (13) are satisfied. Furthermore, **Problem 1** is feasible at the next time instant $t_k = \delta$ and $(x(\delta), \delta) \in \{(x, t)|\tilde{V}_T(x(t), t) \le \tilde{V}_{\max}\}$. Thus, recursive feasibility and inequalities (12) and (13) can be derived by induction.

(14) leads to that if $\|e_p(t_k)\| > \tilde{c}_p$,

$$\tilde{V}_T(x(t_{k+1}), t_{k+1}) \le -\xi_{T,T'} + \tilde{V}_T(x(t_k), t_k),$$

implying that $\|e_p(t_k)\|$ should be less or equal to $\tilde{c}_p$ within $\lceil \frac{\tilde{V}_{\max} - \alpha_\delta(c_p)}{\xi_{T,T'}} \rceil$ steps and by Theorem 3.1, $(x(t_k), t_k) \in \mathbb{S}^\delta_{c_p}$ for all t_k thereafter.

Theorem 3.3 *Let* $T > \tilde{T}_1$, *and Assumptions 2.2, 3.1, 3.2 and 3.3 hold. Then the average performance satisfies that*

$$\lim_{K\to\infty} \frac{1}{K\delta} \int_0^{K\delta} l(x(t), u(t), t)dt \le \frac{V_{T_p,min}}{T_p} + \frac{\tilde{\theta}(T - \delta)}{\delta}, \quad K \in \mathbb{N}.$$

PROOF. We first consider the case when $(x(0), 0) \in \mathbb{S}^\delta_{c_p}$. By invoking (11) for states along the trajectory of the closed-loop at all time instants $i\delta$, $i \in \mathbb{N}$ we have

$$\begin{aligned} &\tilde{V}_T(x(t_{k+1}), t_{k+1}) - \tilde{V}_T(x(t_k), t_k) \\ \le& -\int_{t_k}^{t_{k+1}} l(x(s), u(s), s)ds + \int_{t_k}^{t_{k+1}} l(x_p(s), u_p(s), s)ds \\ &+ \lambda(t_{k+1}, x(t_{k+1})) - \lambda(t_k, x(t_k)) \end{aligned}$$

Summing it from $k = 0$ to $K - 1$ results in

$$\begin{aligned} &\tilde{V}_T(x(t_K), t_K) - \tilde{V}_T(x(0), 0) \\ \le& -\int_0^{t_K} l(x(s), u(s), s)ds + \int_0^{t_K} l(x_p(s), u_p(s), s)ds \\ &+ K\tilde{\theta}(T - T') + \lambda(t_K, x(t_K)) - \lambda(0, x(0)), \end{aligned}$$

leading to

$$\begin{aligned} &\frac{\tilde{V}_T(x(t_K), t_K) - \tilde{V}_T(x(0), 0)}{K\delta} \\ \le& -\frac{1}{K\delta}\int_0^{t_K} l(x(s), u(s), s)ds + \frac{1}{K\delta}\int_0^{t_K} l(x_p(s), u_p(s), s)ds \\ &+ \frac{\tilde{\theta}(T - T')}{\delta} + \frac{\lambda(t_K, x(t_K)) - \lambda(0, x(0))}{K\delta}. \end{aligned}$$

Since $(x(t_k), t_k) \in \mathbb{S}^\delta_{c_p}$ for all $k \in \mathbb{N}$, $\tilde{V}_T(x(t_k), t_k)$ and $\lambda(t_k, x(t_k))$ are finite, we have

$$\lim_{K\to\infty} \frac{1}{K\delta} \int_0^{K\delta} l(x(s), u(s), s)ds$$

$$\le \lim_{K\to\infty} \frac{1}{K\delta} \int_0^{K\delta} l(x_p(s), u_p(s), s)ds + \frac{\tilde{\theta}(T-T')}{\delta}$$
$$= \frac{V_{T_p,\min}}{T_p} + \frac{\tilde{\theta}(T-T')}{\delta},$$

for any $T' > \delta$. Then by the continuity of $\tilde{\theta}$ we can claim that

$$\lim_{K\to\infty} \frac{1}{K\delta} \int_0^{K\delta} l(x(s), u(s), s)ds \le \frac{V_{T_p,\min}}{T_p} + \frac{\tilde{\theta}(T-\delta)}{\delta}.$$

Now we consider the case when $\tilde{V}_T(x(0), 0) \le \tilde{V}_{\max}$. In Theorem 3.2 we have shown that in this case $(x(t_k), t_k)$ will enter $\mathbb{S}^{\delta}_{cp}$ within a finite number of steps. Suppose that $(x(t_k), t_k)$ enters $\mathbb{S}^{\delta}_{cp}$ at $k = K^*$. Then we have

$$\frac{\tilde{V}_T(x(t_K), t_K) - \tilde{V}_T(x(0), 0)}{K\delta}$$
$$\le -\frac{1}{K\delta} \int_0^{t_K} l(x(s), u(s), s)ds + \frac{1}{K\delta} \int_0^{t_K} l(x_p(s), u_p(s), s)ds$$
$$+ \frac{K^* \tilde{\theta}_{\tilde{V}_{\max}}(T-\delta)}{K\delta} + \frac{(K-K^*)\tilde{\theta}(T-\delta)}{K\delta}$$
$$+ \frac{\lambda(t_K, x(t_K)) - \lambda(0, x(0))}{K\delta},$$

for $K > K^*$. Taking $K \to \infty$ yields the desired result.

Remark 3.3 *Theorem 3.3 shows that the difference between the average tracking error of the closed-loop trajectory and the optimal reachable one is upper bounded by $\frac{\tilde{\theta}(T-\delta)}{\delta}$. For a fixed sampling period δ, by increasing T, the difference decreases, which is consistent with our intuition that longer prediction leads to better closed-loop performance. On the other hand, by letting $\delta \to 0$, we can consider the set $\mathbb{S}^{\delta}_{c_p} = \{(x, t) | \tilde{V}_T(x, t) \le \alpha_\delta(c_p)\}$. According to Theorems 3.2, $(x(t_k), t_k)$ will enter $\mathbb{S}^{\delta}_{cp}$ with in a finite number of steps and stays inside thereafter. Note that when $\delta \to 0$, the set $\mathbb{S}^{\delta}_{c_p}$ shrinks to the optimal reachable trajectory itself. Therefore, the average tracking performance becomes the same as that of the optimal reachable trajectory. However, in this case, when $\delta \to 0$, T should go to infinity according to Theorems 3.1 and 3.2.*

Remark 3.4 *In Theorem 3.3, we can see that when $\delta \to 0$, T has to tend to infinity such that the average performance is bounded. This is due to the fact that (10) is not tight. For this inequality, when $\delta \to 0$, the left hand side tends to zero while the right hand side tends to $\tilde{\theta}(T)$, which is a positive constant. One reason is that the rotated cost $\tilde{V}_T$ is used to show the stability while the control input applied to the system is derived from the original cost V_T. Different from setpoint stabilization, the rotated MPC problem is not equivalent to the original one. As a consequence, an auxiliary trajectory combining solutions from the rotated problem and the original problem is used for analysis. The use of such a mixed virtual trajectory leads to some overestimate of the discrepancy. The other reason is that the turnpike property only tells us the existence of time instants when $\|e^*_p\|$ and $\|\tilde{e}^*_p\|$ are small enough. However, without terminal constraint, it is hard to say more about those time instants, for example, if they appear at the end of the prediction horizon or not. Therefore, in the proof of Theorem 3.1, we cannot precisely characterize the key parameter t^* and have to estimate it for the worst case. Although the first one can be reduced by assuming that the system dynamics and the storage function are Lipschitz continuous, the second one seems to be much more difficult to improve.*

Remark 3.5 *In Theorems 3.1 and 3.2 we derive lower bounds of the prediction horizon but their explicit forms are unavailable in general since they depend on some functions which exist but are unknown, e.g., $\gamma_\lambda(\cdot)$. Therefore, Theorems 3.1 and 3.2 just show the existence of such a sufficiently long prediction horizon to ensure practical stability.*

4 Numerical Example

The example in this section is implemented with ICLOCS [21] and solved by IPOPT [22].

We consider the system from [23]

$\dot{x}(t) = Ax(t) + g(x(t)) + Bu(t),$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0.5 \\ -2 \end{bmatrix}$ and $g(x(t)) = \begin{bmatrix} 0 \\ -0.25x_2^3 \end{bmatrix}$.

The control constraint is $-2 \le u(t) \le 2$, $\forall t \ge 0$ and the state constraint is $\begin{bmatrix} -1.5 \\ -1.5 \end{bmatrix} \le \begin{bmatrix} x1 \\ x2 \end{bmatrix} \le \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}$.

The reference trajectory $x_r(t)$ is given by $x_{r,1} = sin(ct)$ and $x_{r,2} = cos(ct)$ where $c = \frac{2\pi}{8}$. By solving **Problem 2** we can see that this trajectory cannot be perfectly tracked and obtain the optimal reachable trajectory x_p. In this case, we apply an MPC controller defined by **Problem 1** to implicitly track the optimal reachable trajectory x_p, which is unknown to the controller.

The stage cost is defined as $l(x(t), u(t), t) = \|x(t) - x_r(t)\|^2$. It has been shown in [23] that $u(t) = v(t) + K(x(t) - x_p(t))$ with $K = [-1.3693\ 5.1273]$ can stabilize the system to the reachable trajectory $x_p(t)$. We further verify that it satisfies Assumptions 2.2 and 3.1 with $V_\epsilon(x, x_p) = \frac{1}{2}\|x - x_p\|^2$, $c_{\epsilon,l} = c_{\epsilon,u} = 0.5$, $k_{\max} = 5.307$, $\rho = 0.8314$, $\epsilon_{p,\text{ref}} = 0.055$.

The sampling period and the prediction horizon are chosen as 0.1 and 2, respectively. In Fig. 1 the reference (unreachable) trajectory x_r, the optimal reachable trajectory x_p and the closed-loop trajectory x are plotted. We can see that the system can almost perfectly track the optimal reachable trajectory even though it is not explicitly known to the controller.

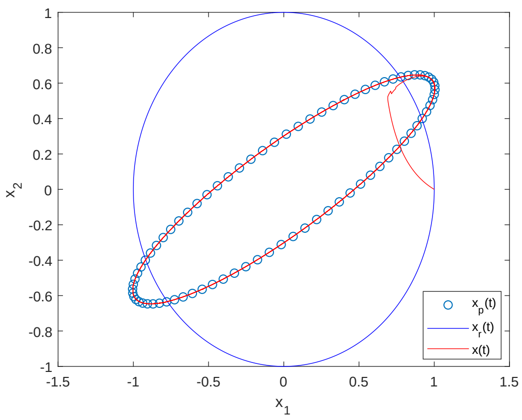

Fig. 1. Tracking the unreachable reference (practically track the optimal reachable reference)

5 Conclusion

In this work, we have extended the unconstrained MPC from setpoint stabilization to dynamic reference tracking for continuous-time nonlinear systems. Compared with most existing tracking MPC, the proposed algorithm does not require offline design of terminal costs or terminal sets. In particular, we focus on the case when the reference trajectory is not perfectly trackable. For such case, the technique for EMPC without terminal conditions in [14] and [13] has been extended to continuous-time cases and the practical tracking stability with respect to the optimal reachable reference has been proved even though the optimal reachable reference is not explicitly known by the controller. Compared with the discrete-time case [13], we have shown that how the required prediction horizon T is related to the sampling period δ, which provides guidance on practical implementations for continuous-time systems.

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