Spin Hall effect in a simple classical picture of spin forces

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Abstract

Spin Hall effect (SHE) in a 2D-Rashba system has been treated in the spin-dependent precession [*J. Sinova et al., Phys. Rev. Letts. 92, 126603 (2004)*.] and the time-space gauge [*T. Fujita et al., New J. Phys. 12, 013016 (2010).*] approaches, both yield SHE conductivity of $\sigma_y^z = \frac{e}{8\pi}$. Separate studies based on the concept of spin transverse force provide a heuristic but not a quantifiable indication of SHE. In this paper, we provide a more complete description of the SHE using the spin orbit gauge approach, unifying σ_y^z under the classical notion of forces and accelerations. Central to this paper is the spin force equations that are satisfied by both σ_y^z and σ_x^z . The idea of spin transverse force having a direct effect on σ_y^z can thus be dismissed. By linking σ_y^z to the spin force, one can construct a simple classical Lorentz force picture to describe the physics of SHE in a 2D-Rashba system.

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Introduction

Following the first description of the spin Hall effect (SHE) by M.I. Dyakonov and V.I. Perel **[1]**, and an early experimental indication of such effect **[2]**, scientific interest in SHE took a long silence before reemerging only in the 21st century with numerous experimental verifications **[3-5]** and theoretical discussions **[6-8]**. SHE was discussed in various systems e.g. the Luttinger, Rashba, cubic and linear Dresselhaus and others with underlying spin orbit coupling. In the Rashba two-dimensional-electron-gas (2DEG) system in particular, SHE was described by Sinova et al. **[9]** with a theory of electron precession that is correlated to its transverse separation, leading to a universal SHE conductivity of $\sigma_y^z = e/8\pi$. A few conclusions in this theory seem to also find support in separate studies. For example, the need for a momentum change $\left(\frac{dp_x}{dt}\right)$ means that impurities scattering might destroy SHE. This is consistent with Inoue's **[10]** prediction that SHE is destroyed by excessive impurities scattering (vertex correction) in a diffusive system. It had also been reasoned that impurities scattering is equivalent to a retardation force which results in average zero momentum change which might thus destroy SHE. The above collectively suggest that spin Hall in a parabolic energy, 2D-Rashba system can only exist with finite carrier acceleration. Recently, T. Fujita et al. **[11,12]** uses the gauge theoretic approach which makes explicit the role of a derivative magnetic field in time-space **[13]** on SHE. Assuming the adiabatic relaxation of electron spin to the vector sum of the derivative field (vertical) and the Rashba spin orbit field (in-plane) leads to the same universal $\sigma_y^z = e/8\pi$. Since the special magnetic field required here depends on $(\frac{dp_x}{dt})$, once again the crucial role of momentum change (electric field) is necessary. On the other hand, a separate body of work **[14-17]** which studies the spin-dependent transverse force in terms of the spin orbit gauge (classical Yang-Mills) seems to provide a rather heuristic indication of SHE, but not a quantifiable conductivity. The spin transverse force picture has predicted numerous precession correlated electron motion including Zitterbewegung.

We will show in this paper that the spin orbit gauge approach **[14-17]** can provide a more complete description of the SHE in terms of the spin forces, unifying σ_y^z under the classical notion of forces and accelerations. Thus, central to this paper is the derivation of the spin force equations that are satisfied by both σ_y^z and σ_x^z . One derives, with direct participation of the spin orbit gauge, the equations of motion that connect σ_y^z and σ_x^z to respectively, the spin longitudinal and the spin transverse forces. In SHE context, a link between σ_y^z and the spin longitudinal force has thus been established, which also dismisses the simple notion that σ_y^z is directly linked to the spin transverse force. With the link established through the spin force equations, one can now focus on σ_y^z and derive its explicit expression using microscopic approaches that can also be found in Refs **[9, 10, 12, 13].** Thus, the explicit quantum mechanical expression of σ_y^z remain valid under the classical framework of forces.

A. Spin Force Equations for a Rashba system

We will begin with the Hamiltonian (vector in bold notation) of a Rashba SOC system in the presence of an external electric field:

$$H = \frac{p^2}{2m} + \lambda\, \boldsymbol{\sigma}.(\boldsymbol{p} \times \boldsymbol{z}) + e\boldsymbol{E_a}.\boldsymbol{r}$$

(1)

where one has $\lambda = \frac{\alpha_R}{\hbar}$, and α_R is the Rashba coupling constant measured in unit of eVm, and E_a is the external electric field. It is worth noting that λ has the dimension of velocity, i.e. $[\lambda] = meter/second$. Thus when written in $H = \gamma\, \boldsymbol{\sigma}.\frac{\lambda}{\gamma}(\boldsymbol{p} \times \boldsymbol{z})$, one has an effective magnetic field of $\boldsymbol{b} = \frac{\lambda}{\gamma}(\boldsymbol{p} \times \boldsymbol{z})$ where $\boldsymbol{b}$ has the dimension $[b] = Tesla$, and $[\gamma] = Joule/Tesla$. In the language of gauge theory, the Rashba SOC energy is equivalent to the presence of a non-Abelian gauge potential of $\boldsymbol{A} = \frac{m\lambda}{e}(\sigma_y, -\sigma_x, 0)$, so that up to the first order in SOC constant, the Hamiltonian now reads $H = \frac{1}{2m}(\boldsymbol{p} - e\boldsymbol{A})^2 + e\boldsymbol{E_a}.\boldsymbol{r}$. The curvature of the gauge potential has the physical meaning of an effective field (magnetic or electric) given by $F_{\mu\nu} = \frac{ie}{\hbar}[A_\mu, A_\nu]$. The z component of this effective field will be magnetic,

$$B_z\boldsymbol{n_z} = F_{xy}\boldsymbol{n_z} = -\frac{2m^2\lambda^2}{e\hbar}\sigma_z\boldsymbol{n_z}$$

(2)

in which $\boldsymbol{n_z}$ is vertical unit vector. Due to the σ_z coefficient, this effective magnetic field can be treated like it is spin-dependent, i.e. it points up (down) when "seen" by electron of spin down (up). This field, coupled with classical electron motion, thus generates Lorentz-like force acting on the moving electrons. For example, if a spin-down electron initially moving along x-direction, it will feel a transverse force and move to the left, likewise, spin-up will move to the right, leading to the separation of spins and hence

$$\boldsymbol{f^{YM}} = eB_z\boldsymbol{v} \times \boldsymbol{n_z}$$

(3)

The above is a "force" expression which corresponds to the Rashba spin orbit gauge. Based on the same classical notion of $\boldsymbol{f^{YM}} = e\boldsymbol{v} \times \boldsymbol{B}$, where $\boldsymbol{B}$ is a non-Abelian effective magnetic field, the general expression corresponding to an arbitrary non-Abelian gauge $\boldsymbol{A}$ is

$$f_i^{YM} = \frac{ie^2}{m\hbar}\left(P_\mu + eA_\mu\right)\left[A_i, A_\mu\right]$$

(4)

The physics is illustrated in Fig.1 which shows a Lorentz-force effect on electron of one spin orientation.

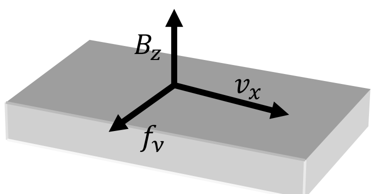

Fig.1. The picture of spin transverse force experienced by electron traveling along x at a constant velocity.

The above indeed gives a heuristic, and intuitive picture of spin separation, but the actual dynamics of electron remains unclear. For example, electron spin undergoes precession about the spin-orbit effective magnetic field $\boldsymbol{b}$ (not gauge curvature $\boldsymbol{B}$). As a result the gauge curvature $\boldsymbol{B}$ acting on each electron also flips up or down at the same rate, leading to the transverse oscillation of electrons. This may lead to one arguing that the spin transverse Lorentz force alone could not have sufficiently explained the separation of spin in SHE.

We will now show, as described earlier that the spin orbit gauge could lead to a set of spin force equations which relate electron acceleration due to $\frac{dp_x}{dt}$ to spin conductivity of σ_y^z and σ_x^z. We start with the classical equation of motion for force i.e. $f = m\frac{dv}{dt}$. Since in classical mechanics, $= \frac{\partial H}{dp}$, one can rewrite the force equation but now taking $\boldsymbol{v}$ as an operator

$$\boldsymbol{f} = m\frac{d\langle \boldsymbol{v}\rangle}{dt} = \frac{d\langle \boldsymbol{p} - e\boldsymbol{A}\ \rangle}{dt}$$

(5)

Applying the quantum dynamic of $i\hbar \frac{d\langle \boldsymbol{O} \rangle}{dt} = \langle\, [\, \boldsymbol{O}, H\,]\, \rangle + i\hbar \langle \frac{\partial \boldsymbol{O}}{\partial t} \rangle$ to the velocity operator, one arrives at

$$\boldsymbol{f} = \left(\frac{1}{i\hbar} \langle [\boldsymbol{p}\,, H] \rangle + \langle \frac{\partial \boldsymbol{p}}{\partial t} \rangle \right) + \left(-\frac{e}{i\hbar} \langle\, [\, \boldsymbol{A}\,, H\,]\, \rangle - e \langle \frac{\partial \boldsymbol{A}}{\partial t} \rangle \right) \equiv \boldsymbol{f_1} + \boldsymbol{f_2}$$

(6)

where $\boldsymbol{A} = G(\, \boldsymbol{\sigma} \times \boldsymbol{E_R})$, one has $\langle \frac{\partial \boldsymbol{A}}{\partial t} \rangle = G \langle\, \boldsymbol{\sigma} \times \frac{\partial \boldsymbol{E}}{\partial t} + \frac{\partial \boldsymbol{\sigma}}{\partial t} \times \boldsymbol{E_R} \,\rangle$ and $\boldsymbol{E_R}$ is the electric field associated with the Rashba effect. For physical clarity, we note that $G = \frac{m\lambda}{eE_R}$ would have the dimension of $[G] = time$ one recalls λ to have the dimension of velocity. Thus,

$$\boldsymbol{f_1} = \frac{1}{i\hbar} \langle \left[\boldsymbol{p}, \left(\frac{p^2}{2m} + \frac{e\,(\boldsymbol{p}.\boldsymbol{A} + \boldsymbol{A}.\boldsymbol{p})}{2m} + e\, \boldsymbol{E}.\boldsymbol{r} \right) \right] \rangle = \frac{e}{i\hbar} \langle \left[\boldsymbol{p}\,, \frac{(\boldsymbol{p}.\boldsymbol{A} + \boldsymbol{A}.\boldsymbol{p})}{2m} + \boldsymbol{E}.\boldsymbol{r} \right] \rangle$$

(7a)

$$\boldsymbol{f_2} = \frac{-e}{i\hbar} \langle \left[\boldsymbol{A}\,, \left(\frac{p^2}{2m} + \frac{e\,(\boldsymbol{p}.\boldsymbol{A} + \boldsymbol{A}.\boldsymbol{p})}{2m} + e\, \boldsymbol{E}.\boldsymbol{r} \right) \right] \rangle = \frac{ie}{\hbar} \langle \left[\boldsymbol{A}\,, \frac{e(\boldsymbol{p}.\boldsymbol{A} + \boldsymbol{A}.\boldsymbol{p})}{2m} \right] \rangle$$

(7b)

For consistency, the longitudinal direction is always taken to be x, the transverse direction y. As the term $\left[\boldsymbol{p}\,, \frac{(\boldsymbol{p}.\boldsymbol{A}+\boldsymbol{A}.\boldsymbol{p})}{2m} \right]$ vanishes when the spin orbit constant G is spatially uniform, which is assumed in this paper to simplify analysis, $\boldsymbol{f_1}$ is reduced to pure electrical force, i.e. $\boldsymbol{f_1} = e\boldsymbol{E}$. This is true regardless of the quantum state of the carrier because $\boldsymbol{E}$ is not an operator. Thus, referring back to Eq.(5) one could see that $\frac{d\langle \boldsymbol{p} \rangle}{dt} = e\boldsymbol{E}$**,** which physically means that external electrical force has a direct effect on the electron's momentum but not necessaily its velocity. Although in 2DEG, or metallic system, velocity is a linear function of the momentum, this would not be the case in other non-parabolic energy system. For a general spin orbit gauge $\boldsymbol{A}$, one can obtain with $\boldsymbol{f_2} = \frac{ie}{\hbar} \langle \left[\boldsymbol{A}\,, \frac{e(\boldsymbol{p}.\boldsymbol{A}+\boldsymbol{A}.\boldsymbol{p})}{2m} \right] \rangle$, an expression of

$$f_i = \frac{ie^2}{m\hbar} \langle\, P_\mu \left[A_i\,, A_\mu \right] \rangle$$

(8)

which resembles the classical Yang-Mills force of Eq.(4). Combining $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$**,** and ignoring the higher order terms, the spin force equations can be written as

$$m\frac{d\langle v_i\rangle}{dt} = eE_i + f_i^{YM}$$

(9)

Thus, the heuristic Yang-Mills force has been fitted, with approximations, into the classical EOM of acceleration in the Heisenberg picture, providing a clearer picture of how Yang-Mills force generates electron motion. Applying $\boldsymbol{f_2}$ to the special case of the Rashba spin orbit gauge $\boldsymbol{A} = \frac{m\lambda}{e}(\sigma_y, -\sigma_x, 0)$, one produces

$$\boldsymbol{f_2} = \frac{m\lambda^2}{\hbar}\langle \{\sigma_z, p_y\}\,\boldsymbol{i} - \{\sigma_z, p_x\}\,\boldsymbol{j}\rangle$$

(10)

The Rashba $\boldsymbol{A}$ above has generated a pair of perpendicular spin forces. But what is of physical importance is the fact that each force component is perpendicular to a momentum coupled to it. For example, the term $\{\sigma_z, p_y\}\,\boldsymbol{i}$ contains p_y. One can thus say that a longitudinal force is coupled to a transverse spin current. Combining $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$**,** the spin force equations are

$$m\frac{d\langle v_x\rangle}{dt} = eE_x + \frac{m\lambda^2}{\hbar}\langle \{\sigma_z, p_y\}\rangle$$

(11a)

$$m\frac{d\langle v_y\rangle}{dt} = eE_y - \frac{m\lambda^2}{\hbar}\langle \{\sigma_z, p_x\}\rangle$$

(11b)

The spin current operator is,

$$\boldsymbol{j}^z = \frac{\hbar}{4}\{\boldsymbol{v}, \sigma_z\} = \frac{\hbar}{4m}\{(\boldsymbol{p} - e\boldsymbol{A}), \sigma_z\},$$

(12)

where spin current has the dimension of $[\boldsymbol{j}^z] = angular\ momentum/second$. One notes that $\boldsymbol{A} = \frac{m\lambda}{e}(\sigma_y, -\sigma_x, 0)$ vanishes by virtue of $\{\boldsymbol{A}, \sigma_z\} = 0$, thus resulting in $\boldsymbol{j}^z = \frac{\hbar}{4m}\{\boldsymbol{p}, \sigma_z\}$. One further deduces that Rashba $\boldsymbol{A}$ is not directly related to spin conductivity of σ_y^z. In summary for Rashba $\boldsymbol{A}$, spin forces are coupled to the spin current as follows:

$$\begin{pmatrix} j_x^z \\ j_y^z \end{pmatrix} = \frac{\hbar^2}{4m^2\lambda^2}\begin{pmatrix} f_{2y} \\ f_{2x} \end{pmatrix},$$

(13)

Therefore, the SHE current is related to the spin forces transverse to its direction. And one notes by inspection that the SHE current j_y^z is related to the spin longitudinal force but not the spin transverse force. With the spin force equations and the notion of classical Yang-Mills field (gauge curvature) acting like a spin magnetic field (B_z), one can construct a simple classical Lorentz force picture to describe the physics of SHE in a 2D-Rashba system as shown in Fig.2 below.

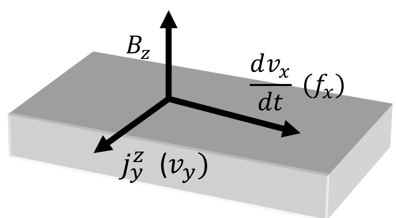

Fig.2. A modified Lorentz force picture which describes SHE current as related to the spin longitudinal force.

Compared to the charge Hall effect where $f_y = B_z q v_x$ (see Fig.1), the spin Lorentz force picture is a rotated version (about z axis over an azimuthal angle) of Fig.1. In summary, the spin force equation of motion for the Rashba $\boldsymbol{A}$

$$m\frac{d\langle v_i\rangle}{dt} = eE_i + \frac{4m^2\lambda^2}{\hbar^2}\langle\psi_{ip}|\, j_j^z|\psi_{ip}\rangle\,\varepsilon_{ijz} \tag{14}$$

Note that $|\psi_{ip}\rangle$ is the in-plane eigenstate of the Rashba 2D system, or the spin eigensolution of Eq.(1). As mentioned earlier that once the link is established through the above, one can continue to derive the explicit expression of σ_y^z or j_y^z, noting that $j_y^z = \sigma_y^z E_x$. The explicit expression of the SHE current will depend on the quantum state of the carrier i.e. $|\psi_{ip}\rangle$, which can be obtained by solving the Hamiltonian. There are many ways one can derive the explicit SHE current, but all explicit quantum mechanical expression of SHE current remain valid under the semiclassical framework of forces and accelerations of Eq.(14).

B. Spin Force Equations for a Rashba System with Time-gauge

The SHE current as shown in Eq.(14), $\langle\psi_{ip}(t)|j_y^z(t)|\psi_{ip}(t)\rangle$ vanishes because the spin state of electron $|\psi_{ip}(t)\rangle$ in Rashba 2DEG is generally in-plane, while the spin current operator contains a Pauli-z matrix. Although we know that in reality, spin tilts out-of-plane as electron experiences a momentum change of

($\frac{dp_x}{dt}$) **[11, 12]**, this effect is not reflected in the Hamiltonian. However, the momentum change effect is discussed in Ref.11 and 12 using the gauge approach of electric curvature. The Hamiltonian used in Eq.(1) is the time-independent form (H_S). One could check that the Heisenberg SHE current $j_y^z(t) = e^{iH_st} j_y^z e^{-iH_st}$ evolves with the generator H_S. But this will not produce the effect of momentum change or electric curvature. On the other hand, the eigenstate of H_S which is $|\psi_{ip}(t)\rangle$ will only evolve in dynamic phase. Therefore, to obtain non-vanishing SHE current, one needs the time-dependent treatment **[11, 12]**, where an electric curvature (effective magnetic field) appears vertical to the 2D plane via a local gauge transformation in momentum space-time. A sum field ($\boldsymbol{\Omega}$) which is the vector sum of the vertical time-field $\frac{\hbar(\partial_t \boldsymbol{b}\times\boldsymbol{b})}{\gamma b^2}$ and the in-plane spin orbit field ($\boldsymbol{b}$) could thus be defined. Electron with spin aligned parallel/antiparallel to the total field is identified with the $\rho = \pm$ bands. In summary, one ventures from the original H_S to a time-dependent $H_I(t)$, and performs the local gauge transformation. An inverse rotation would yield the Hamiltonian with the electric curvature. In the accelerating frame ($\frac{dp_x}{dt}$) of electron, one obtains back a time-independent Hamiltonian of

$$H_s^f = \frac{p^2}{2m} + \gamma\, \boldsymbol{\sigma}.\boldsymbol{b} + \hbar\boldsymbol{\sigma}.\boldsymbol{a} + e\boldsymbol{E}.\boldsymbol{r} \tag{15}$$

where $\boldsymbol{a}$ is perpendicular to $\boldsymbol{b}$, and the explicit expression is $\hbar\boldsymbol{a} = \left(\frac{\hbar(\partial_t \boldsymbol{b}\times\boldsymbol{b})}{b^2}\right) = \frac{\hbar e E_x p_y}{p^2}\boldsymbol{z}$. We note in passing that two rotations have been performed to arrive at Eq.(13) see Ref **[11, 12]** for details. In fact, noting that $\boldsymbol{b} = \frac{\lambda}{\gamma}(\boldsymbol{p}\times\boldsymbol{z})$, one could reason with classical physics that as spin aligns to the sum field $\boldsymbol{\Omega}$, a vertical component of spin polarization $\langle \boldsymbol{s}\rangle = \rho s\, \boldsymbol{\Omega}/\Omega$ is generated, s is the spin quantum number. Now, it can be derived, noting $= \alpha_R/\hbar$, and using the time-independent H_s^f that the modified gauge potential is:

$$\boldsymbol{A} = \frac{m\lambda}{e}(\sigma_y\, , -\sigma_x + \frac{\hbar e E_x}{p^2\lambda}\sigma_z\, ,0) \tag{16}$$

Note that the momentum p is in-plane. With Eq.7(b), the spin force equations for the Rashba time gauge system is

$$m\frac{d\langle v_x\rangle}{dt} = eE_x + \frac{m\lambda^2}{\hbar}\langle\psi_\rho|\{\sigma_z\, , p_y\}|\psi_\rho\rangle + \frac{eE_x m\lambda}{p^2}\langle\psi_\rho|\{\sigma_x\, , p_y\}|\psi_\rho\rangle$$
$$m\frac{d\langle v_y\rangle}{dt} = eE_y - \frac{m\lambda^2}{\hbar}\langle\psi_\rho|\{\sigma_z\, , p_x\}|\psi_\rho\rangle - \frac{eE_x m\lambda}{p^2}\langle\psi_\rho|\{\sigma_x\, , p_x\}|\psi_\rho\rangle$$

(17)

The SHE current of $j_\mu^z = \frac{\hbar}{4}\langle\{v_\mu , \sigma_z\}\rangle = \frac{\hbar}{4m}\langle\psi_\rho|\{(p_\mu - eA_\mu) , \sigma_z\}|\psi_\rho\rangle$ can, once again, be fitted with approximations into the spin force equations, in the time gauge approach. Different from the original Rashba system, in the Rashba time-gauge system, the expression $\frac{m\lambda^2}{\hbar}\langle\psi_\rho|\{\sigma_z , p_y\}|\psi_\rho\rangle$ is non-vanishing. In fact it can be derived from the polarization expression of $\langle \boldsymbol{s}\rangle = \rho s\, \boldsymbol{\Omega}/\Omega$, which is non-vanishing, and finite. Therefore, the SHE current will be non-vanishing consistent with previous accounts.

Conclusion

In summary, SHE conductivity of σ_y^z or j_y^z had been fitted into the macroscopic spin force equations, providing an intuitive Lorentz-force picture description of the SHE, which is a rotated version of the Lorentz force picture for ordinary charge Hall. Explicit quantum mechanical expression of σ_y^z or j_y^z can be derived separately. Therefore, what is provided in this paper is a more complete description of the SHE using the spin orbit gauge approach, unifying σ_y^z or j_y^z under the classical notion of forces and accelerations. The idea of spin transverse force having a direct effect on σ_y^z is also dismissed.

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