SUPPLEMENTARY DERIVATIONS ON

"ACCURATE IMU FACTOR USING SWITCHED LINEAR SYSTEMS FOR VIO"

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1 Preliminaries

Let $\theta(=\begin{bmatrix}\theta_1 & \theta_2 & \theta_3\end{bmatrix}^T)$ be a vector in the set $\mathbb{R}^3$, and a skew symmetric matrix $\theta^\wedge \in \mathbb{R}^{3\times 3}$ is defined as

$$\theta^\wedge = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}. \tag{1}$$

It can be easily derived that

$$\begin{cases} (\theta^\wedge)^2 = \begin{bmatrix} -\theta_2^2-\theta_3^2 & \theta_1\theta_2 & \theta_1\theta_3 \\ \theta_1\theta_2 & -\theta_1^2-\theta_3^2 & \theta_2\theta_3 \\ \theta_1\theta_3 & \theta_2\theta_3 & -\theta_1^2-\theta_2^2 \end{bmatrix} \\ (\theta^\wedge)^3 = -\|\theta\|^2\theta^\wedge \end{cases}, \tag{2}$$

where $\|\theta\|^2$ is $(\theta_1^2+\theta_2^2+\theta_3^2)$.

A useful property of the skew matrices

$$a^\wedge b = -b^\wedge a \tag{3}$$

Table 1: Coefficients of matrices $E(\theta)$, $\Gamma(\theta)$, $\Lambda(\theta)$ and $\Phi(\theta)$

Matrix	Coefficient of I	Coefficient of $\theta^\wedge$	Coefficient of $(\theta^\wedge)^2$
$E(\theta)$	1	$\frac{h_1(\|\theta\|)}{\|\theta\|}$	$\frac{h_2(\|\theta\|)}{\|\theta\|^2}$
$\Gamma(\theta)$	1	$\frac{h_2(\|\theta\|)}{\|\theta\|^2}$	$\frac{h_3(\|\theta\|)}{\|\theta\|^3}$
$\Lambda(\theta)$	$\frac{1}{2}$	$\frac{h_3(\|\theta\|)}{\|\theta\|^3}$	$\frac{h_4(\|\theta\|)}{2\|\theta\|^4}$
$\Phi(\theta)$	$\frac{1}{6}$	$\frac{h_4(\|\theta\|)}{2\|\theta\|^4}$	$\frac{h_5(\|\theta\|)}{6\|\theta\|^5}$

Four matrices $E(\theta)$, $\Gamma(\theta)$, $\Lambda(\theta)$ and $\Phi(\theta)$ are then defined as in Table 1 with the functions $h_i(z)(i=1,2,3,4,5)$ being defined as

$$\begin{cases} h_1(z) = \sin z \\ h_2(z) = 1-\cos z \\ h_3(z) = z-\sin z \\ h_4(z) = 2\cos z - 2 + z^2 \\ h_5(z) = 6\sin z - 6z + z^3 \end{cases}. \tag{4}$$

These four matrices will be used to build up an IMU integration model for the proposed VIO algorithm. The first order approximation of $E(\theta)$, $\Gamma(\theta)$, $\Lambda(\theta)$ and $\Phi(\theta)$ are given as

$$\begin{cases} E(\theta) \approx I_{3\times 3} + \theta^\wedge \\ \Gamma(\theta) \approx I_{3\times 3} + \frac{1}{2}\theta^\wedge \\ \Lambda(\theta) \approx \frac{I_{3\times 3}}{2} + \frac{1}{6}\theta^\wedge \end{cases}, \tag{5}$$

and they will be utilized to study affect of noise and bias.

SO(3) is the group of 3-D rotation matrices and it is formally defined as

$$SO(3) \doteq \{R \in \mathbb{R}^{3\times 3}, R^T R = I_{3\times 3}, det(R) = 1\}. \tag{6}$$

The group operation is the conventional matrix multiplication, and the inverse is the conventional matrix transpose. The group SO(3) is a smooth manifold. The tangent space to the manifold (at the identity) is represented by so(3), which is usually called the *Lie algebra* and is the same as the space of 3×3 skew symmetric matrices. Let R be a matrix in the set SO(3). It can be derived that

$$\begin{cases} E(R\theta) = RE(\theta)R^T \\ \Gamma(R\theta) = R^T\Gamma(\theta)R \\ \Lambda(R\theta) = R^T\Lambda(\theta)R \end{cases}. \tag{7}$$

Another exponential map property that will be used later is

$$E(\theta)R = RE(R^T\theta) \tag{8}$$

Using the first-order approximation, it can be obtained that

$$\begin{cases} E(\theta+\delta\theta) \approx E(\theta)E(\Gamma(-\theta)\delta\theta) \\ \Gamma(\theta+\delta\theta) \approx \Gamma(\theta)\Gamma(\Lambda(-\theta)\delta\theta) \\ \Lambda(\theta+\delta\theta) \approx 2\Lambda(\theta)\Lambda(\Phi(-\theta)\delta\theta) \end{cases}. \tag{9}$$

Defining an operation $Log(\cdot)$ as [1]

$$Log(E(\theta)) = \theta, \tag{10}$$

it can be derived that

$$Log(E(\theta)E(\delta\theta)) \approx \theta + \Gamma^{-1}(\theta)\delta\theta \ , \tag{11}$$

where $\Gamma^{-1}(\theta)$ is given in (12)

$$\Gamma^{-1} = I_{3\times 3} - \frac{h_2(\|\theta\|)(h_1(\|\theta\|) + h_3(\|\theta\|))}{\|\theta\|(h_1^2(\|\theta\|) + h_2^2(\|\theta\|))}\theta^\wedge + \frac{h_2^2(\|\theta\|) - h_1(\|\theta\|)h_3(\|\theta\|)}{\|\theta\|^2(h_1^2(\|\theta\|) + h_2^2(\|\theta\|))}(\theta^\wedge)^2 \tag{12}$$

2 A New IMU Integration Model Using Switched Linear Systems

The new model, has been introduced in [2], can be represented as the following:

$$\begin{cases} P_W(k+1) = P_W(k) + V_W(k)\Delta t \\ \qquad\qquad + (\frac{1}{2}g_W + R_{BW}(k)\Lambda(\theta_B(k))a_B(k))\Delta t^2 \\ V_W(k+1) = V_W(k) + (g_W + R_{BW}(k)\Gamma(\theta_B(k))a_B(k))\Delta t \\ R_{BW}(k+1) = R_{BW}(k)E(\theta_B(k)) \end{cases}, \tag{13}$$

Suppose that two video frames are captured at the time instances $k = i$ and $k = j$. The discrete dynamic model in the equation (13) will be used to integrate all the IMU data between the two video frames as follows:

$$\begin{cases} P_W(j) = P_W(i) + \Theta(i,j) + R_{BW}(i)\zeta(i,j) \\ V_W(j) = V_W(i) + g_W \sum_{k=i}^{j-1} \Delta t + R_{BW}(i)\mu(i,j) \\ R_{BW}(j) = R_{BW}(i)F(i,j) \end{cases}, \tag{14}$$

where the matrix $F(i,j)$, and the vectors $\Theta(i,j)$, $\zeta(i,j)$ and $\mu(i,j)$ are computed as

$$\begin{cases} F(i,j) = \prod_{k=i}^{j-1} E(\theta_B(k)) \\ \Theta(i,j) = V_W(i)\sum_{k=i}^{j-1}\Delta t + \frac{g_W}{2}(\sum_{k=i}^{j-1}\Delta t)^2 \\ \zeta(i,j) = \sum_{k=i}^{j-1}(F(i,k)\Lambda(\theta_B(k))a_B(k)\Delta t^2 + \mu(i,k)\Delta t) \\ \mu(i,j) = \sum_{k=i}^{j-1} F(i,k)\Gamma(\theta_B(k))a_B(k)\Delta t \end{cases}. \tag{15}$$

3 Preintegration Measurements

Define three relative relative motion increments which are independent of the pose and velocity at t_i as

$$\begin{cases} \Delta R_{BW}(i,j) \doteq R_{BW}^T(i)R_{BW}(j) = F(i,j) \\ \Delta V_W(i,j) \doteq R_{BW}^T(i)(V_W(j) - V_W(i) - g\sum_{k=i}^{j-1}\Delta t) = \mu(i,j) \\ \Delta P_W(i,j) \doteq R_{BW}^T(i)(P_W(j) - P_W(i) - \Theta(i,j)) = \zeta(i,j) \end{cases}. \tag{16}$$

The measurements, namely $\tilde{a}_B(k)$, and $\tilde{\omega}_B(k)(k \in [i,j])$, are affected by additive white noise η and a slowly varying sensor bias b as [1]

$$\begin{cases} \tilde{a}_B(k) = a_B(k) + \bar{b}^a(i) + \eta^a(k) \\ \tilde{\omega}_B(k) = \omega_B(k) + \bar{b}^g(i) + \eta^g(k) \end{cases} . \tag{17}$$

Let $(\tilde{a}_B(k) - \bar{b}^a(i))$ and $(\tilde{\omega}_B(k) - \bar{b}^g(i))$ be denoted as $\bar{a}_B(k)$ and $\bar{\omega}_B(k)$, respectively. By respectively replacing $a_B(k)$ and $\omega_B(k)$ by $\bar{a}_B(k)$ and $\bar{\omega}_B(k)$ in Equation (15).

3.1 Preintegrated Rotational Measurements

$$\begin{aligned} \Delta R_{BW}(i,j) &= \prod_{k=i}^{j-1} E(\theta_B(k)) \\ &\overset{\text{eq.(17)}}{=} \prod_{k=i}^{j-1} E((\tilde{\omega}_B(k) - \bar{b}^g(i) - \eta^g(k))\Delta t) \\ &\overset{\text{eq.(9)}}{\approx} \prod_{k=i}^{j-1} E(\bar{\theta}_B(k)) E(-\Gamma(-\bar{\theta}_B(k))\eta^g(k)\Delta t) \\ &\overset{\text{eq.(8)}}{=} \bar{F}(i,j) \prod_{k=i}^{j-1} E(-\bar{F}^T(k+1,j)\Gamma(-\bar{\theta}_B(k))\eta^g(k)\Delta t) \\ &\doteq \bar{F}(i,j) E(-\delta\phi_{BW}^{\eta}(i,j)), \end{aligned} \tag{18}$$

where $\bar{F}(i,j)$ is the preintegrated rotational measurements and $\delta\phi_{BW}^{\eta}$ its noise.

It can be derived that

$$\begin{aligned} \delta\phi_{BW}^{\eta}(i,j) &= -Log\left(\prod_{k=i}^{j-1} E(-\bar{F}^T(k+1,j)\Gamma(-\bar{\theta}_B(k))\eta^g(k)\Delta t)\right) \\ &\overset{\text{eq.(11)}}{\approx} \sum_{k=i}^{j-1} \bar{F}^T(k+1,j)\Gamma(-\bar{\theta}_B(k))\eta^g(k)\Delta t. \end{aligned} \tag{19}$$

3.2 Preintegrated Velocity Measurements

$$
\begin{aligned}
\Delta V_W(i,j) &= \sum_{k=i}^{j-1} F(i,k)\Gamma(\theta_B(k))a_B(k)\Delta t \\
&\overset{\text{eq.(17),(18)}}{\approx} \sum_{k=i}^{j-1} \bar{F}(i,k)E(-\delta\phi_{BW}^{\eta}(i,k))\Gamma(\bar{\theta}_B(k)-\eta^g(k)\Delta t) \\
&\quad (\bar{a}_B(k)-\eta^a(k))\Delta t \\
&\overset{\text{eq.(9)}}{\approx} \sum_{k=i}^{j-1} \bar{F}(i,k)E(-\delta\phi_{BW}^{\eta}(i,k))\Gamma(\bar{\theta}_B(k))\Gamma(-\Lambda(-\bar{\theta}_B(k))\eta^g(k)\Delta t) \\
&\quad (\bar{a}_B(k)-\eta^a(k))\Delta t \\
&\overset{\text{eq.(5)}}{\approx} \sum_{k=i}^{j-1} \bar{F}(i,k)(I_{3\times3}-\delta\phi_{BW}^{\eta}(i,k)^\wedge)\Gamma(\bar{\theta}_B(k))(I_{3\times3}-\frac{1}{2}(\Lambda(-\bar{\theta}_B(k))\eta^g(k)\Delta t))^\wedge) \\
&\quad (\bar{a}_B(k)-\eta^a(k))\Delta t \\
&\approx \sum_{k=i}^{j-1} \Big[\bar{F}(i,k)\Gamma(\bar{\theta}_B(k))\bar{a}_B(k)\Delta t - \bar{F}(i,k)\Gamma(\bar{\theta}_B(k))\eta^a(k)\Delta t \\
&+ \quad \bar{F}(i,k)(-\delta\phi_{BW}^{\eta}(i,k)^\wedge)\Gamma(\bar{\theta}_B(k))\bar{a}_B(k)\Delta t \\
&+ \quad \bar{F}(i,k)\Gamma(\bar{\theta}_B(k))(-\frac{1}{2}(\Lambda(-\bar{\theta}_B(k))\eta^g(k)\Delta t)^\wedge)\bar{a}_B(k)\Delta t\Big] \\
&\overset{\text{eq.(3)}}{=} \sum_{k=i}^{j-1} \bar{F}(i,k)\Gamma(\bar{\theta}_B(k))\bar{a}_B(k)\Delta t \\
&- \quad \sum_{k=i}^{j-1} \bar{F}(i,k)\Gamma(\bar{\theta}_B(k))\eta^a(k)\Delta t \\
&+ \quad \sum_{k=i}^{j-1} \bar{F}(i,k)(\Gamma(\bar{\theta}_B(k))\bar{a}_B(k))^\wedge\delta\phi_{BW}^{\eta}(i,k)\Delta t \\
&+ \quad \frac{1}{2}\sum_{k=i}^{j-1} \bar{F}(i,k)\Gamma(\bar{\theta}_B(k))(\bar{a}_B(k))^\wedge\Lambda(-\bar{\theta}_B(k))\eta^g(k)\Delta t^2 \\
&\doteq \bar{\mu}(i,j)-\delta V_W^{\eta}(i,j),
\end{aligned}
\tag{20}
$$

where $\bar{\mu}(i,j)$ is the preintegrated velocity measurements and $\delta V_W^{\eta}(i,j)$ its noise.

3.3 Preintegrated Position Measurements

$$
\begin{aligned}
\Delta P_W(i,j) \quad &= \quad \sum_{k=i}^{j-1}\left[F(i,k)\Lambda(\theta_B(k))a_B(k)\Delta t^2+\mu(i,k)\Delta t\right] \\
&\overset{\text{eqs.(17),(18),(20)}}{\approx} \quad \sum_{k=i}^{j-1}\left[\bar{F}(i,k)E(-\delta\phi_{BW}^{\eta}(i,k))\Lambda(\bar{\theta}_B(k)-\eta^g(k)\Delta t)\right. \\
&\qquad \left.(\bar{a}_B(k)-\eta^a(k))\Delta t^2+(\bar{\mu}(i,k)+\delta V_W^{\eta}(i,k))\Delta t\right] \\
&\overset{\text{eq.(9)}}{\approx} \quad \sum_{k=i}^{j-1}\left[\bar{F}(i,k)E(-\delta\phi_{BW}^{\eta}(i,k))2\Lambda(\bar{\theta}_B(k))\Lambda(-\Phi(-\bar{\theta}_B(k))\eta^g(k)\Delta t)\right. \\
&\qquad \left.(\bar{a}_B(k)-\eta^a(k))\Delta t^2+(\bar{\mu}(i,k)+\delta V_W^{\eta}(i,k))\Delta t\right] \\
&\overset{\text{eq.(5)}}{\approx} \quad \sum_{k=i}^{j-1}\left[\bar{F}(i,k)(I-\delta\phi_{BW}^{\eta}(i,k)^{\wedge})\Lambda(\bar{\theta}_B(k))(I-\frac{1}{3}(\Phi(-\bar{\theta}_B(k))\eta^g(k)\Delta t))^{\wedge})\right. \\
&\qquad \left.(\bar{a}_B(k)-\eta^a(k))\Delta t^2+(\bar{\mu}(i,k)+\delta V_W^{\eta}(i,k))\Delta t\right] \\
&\approx \quad \sum_{k=i}^{j-1}\left[\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))\bar{a}_B(k)\Delta t^2-\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))\eta^a(k)\Delta t^2\right. \\
&+ \quad \bar{F}(i,k)(-\delta\phi_{BW}^{\eta}(i,k)^{\wedge})\Lambda(\bar{\theta}_B(k))\bar{a}_B(k)\Delta t^2 \\
&+ \quad \bar{F}(i,k)\Lambda(\bar{\theta}_B(k))(-\frac{1}{3}(\Phi(-\bar{\theta}_B(k))\eta^g(k)\Delta t)^{\wedge})\bar{a}_B(k)\Delta t^2 \\
&+ \quad \left.(\bar{\mu}(i,k)+\delta V_W^{\eta}(i,k))\Delta t\right] \\
&\overset{\text{eq.(3)}}{=} \quad \sum_{k=i}^{j-1}\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))\bar{a}_B(k)\Delta t^2+\sum_{k=i}^{j-1}\bar{\mu}(i,k)\Delta t \\
&- \quad \sum_{k=i}^{j-1}\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))\eta^a(k)\Delta t^2 \\
&+ \quad \sum_{k=i}^{j-1}\bar{F}(i,k)(\Lambda(\bar{\theta}_B(k))\bar{a}_B(k))^{\wedge}\delta\phi_{BW}^{\eta}(i,k)\Delta t^2 \\
&+ \quad \frac{1}{3}\sum_{k=i}^{j-1}\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))(\bar{a}_B(k))^{\wedge}\Phi(-\bar{\theta}_B(k))\eta^g(k)\Delta t^3 \\
&+ \quad \sum_{k=i}^{j-1}\delta V_W^{\eta}(i,k)\Delta t \\
&\doteq \quad \bar{\zeta}(i,j)-\delta P_W^{\eta}(i,j), \qquad (21)
\end{aligned}
$$

where $\bar{\zeta}(i,j)$ is the preintegrated position measurements and $\delta P_W^{\eta}(i,j)$ its noise.

4 Iterative Noise Propagation

This section aim to write the preintegrated noise propagation in iterative form. So, the covariance matrix of the preintegrated measurements can be provided which it is crucial and has strong influence on the MAP estimator.

4.1 Preintegrated Rotational Noise

Simply, the last term $(k = j)$ has been taken out of summation of the preintegrated rotational noise $\delta\phi_{BW}^{\eta}$. Accordingly, the preintegrated can be written in iterative way as following:

$$
\begin{aligned}
\delta\phi_{BW}^{\eta}(i,j+1) &\doteq \sum_{k=i}^{j} \bar{F}^T(k+1,j+1)\Gamma(-\bar{\theta}_B(k))\eta^g(k)\Delta t \\
&= \sum_{k=i}^{j-1} \bar{F}^T(k+1,j+1)\Gamma(-\bar{\theta}_B(k))\eta^g(k)\Delta t + \Gamma(-\bar{\theta}_B(j))\eta^g(j)\Delta t \\
&= \bar{F}^T(j,j+1)\sum_{k=i}^{j-1} \bar{F}^T(k+1,j)\Gamma(-\bar{\theta}_B(k))\eta^g(k)\Delta t + \Gamma(-\bar{\theta}_B(j))\eta^g(j)\Delta t \\
&= \bar{F}^T(j,j+1)\underline{\delta\phi_{BW}^{\eta}(i,j)} + \Gamma(-\bar{\theta}_B(j))\Delta t\,\underline{\eta^g(j)} \qquad (22)
\end{aligned}
$$

4.2 Preintegrated Velocity Noise

With repeating same process in (22) for δV_B^η as follows:

$$
\begin{aligned}
\delta V_W^\eta(i,j+1) \quad \doteq \quad & -\sum_{k=i}^{j}\bar{F}(i,k)(\Gamma(\bar{\theta}_B(k))\bar{a}_B(k))^\wedge\delta\phi_{BW}^\eta(i,k)\Delta t \\
- \quad & \frac{1}{2}\sum_{k=i}^{j}\bar{F}(i,k)\Gamma(\bar{\theta}_B(k))(\bar{a}_B(k))^\wedge\Lambda(-\bar{\theta}_B(k))\eta^g(k)\Delta t^2 \\
+ \quad & \sum_{k=i}^{j}\bar{F}(i,k)\Gamma(\bar{\theta}_B(k))\eta^a(k)\Delta t \\
= \quad & -\sum_{k=i}^{j-1}\bar{F}(i,k)(\Gamma(\bar{\theta}_B(k))\bar{a}_B(k))^\wedge\delta\phi_{BW}^\eta(i,k)\Delta t \\
- \quad & \frac{1}{2}\sum_{k=i}^{j-1}\bar{F}(i,k)\Gamma(\bar{\theta}_B(k))(\bar{a}_B(k))^\wedge\Lambda(-\bar{\theta}_B(k))\eta^g(k)\Delta t^2 \\
+ \quad & \sum_{k=i}^{j-1}\bar{F}(i,k)\Gamma(\bar{\theta}_B(k))\eta^a(k)\Delta t \\
- \quad & \bar{F}(i,j)(\Gamma(\bar{\theta}_B(j))\bar{a}_B(j))^\wedge\delta\phi_{BW}^\eta(i,j)\Delta t \\
- \quad & \frac{1}{2}\bar{F}(i,j)\Gamma(\bar{\theta}_B(j))(\bar{a}_B(j))^\wedge\Lambda(-\bar{\theta}_B(j))\eta^g(j)\Delta t^2 \\
+ \quad & \bar{F}(i,j)\Gamma(\bar{\theta}_B(j))\eta^a(j)\Delta t \\
= \quad & \underline{\delta V_W^\eta(i,j)} - \bar{F}(i,j)(\Gamma(\bar{\theta}_B(j))\bar{a}_B(j))^\wedge\Delta t\,\underline{\delta\phi_{BW}^\eta(i,j)} \\
- \quad & \frac{1}{2}\bar{F}(i,j)\Gamma(\bar{\theta}_B(j))(\bar{a}_B(j))^\wedge\Lambda(-\bar{\theta}_B(j))\Delta t^2\underline{\eta^g(j)} \\
+ \quad & \bar{F}(i,j)\Gamma(\bar{\theta}_B(j))\Delta t\,\underline{\eta^a(j)}
\end{aligned}
\tag{23}
$$

4.3 Preintegrated Position Noise

With repeating same process in (22) for δP_B^η as follows:

$$
\begin{aligned}
\delta P_W^\eta(i,j+1) &\doteq -\sum_{k=i}^{j}\bar{F}(i,k)(\Lambda(\bar{\theta}_B(k))\bar{a}_B(k))^\wedge\delta\phi_{BW}^\eta(i,k)\Delta t^2 \\
&- \frac{1}{3}\sum_{k=i}^{j}\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))(\bar{a}_B(k))^\wedge\Phi(-\bar{\theta}_B(k))\eta^g(k)\Delta t^3 \\
&+ \sum_{k=i}^{j}\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))\eta^a(k)\Delta t^2 + \sum_{k=i}^{j}\delta V_W^\eta(i,k)\Delta t \\
&= -\sum_{k=i}^{j-1}\bar{F}(i,k)(\Lambda(\bar{\theta}_B(k))\bar{a}_B(k))^\wedge\delta\phi_{BW}^\eta(i,k)\Delta t^2 \\
&- \frac{1}{3}\sum_{k=i}^{j-1}\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))(\bar{a}_B(k))^\wedge\Phi(-\bar{\theta}_B(k))\eta^g(k)\Delta t^3 \\
&+ \sum_{k=i}^{j-1}\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))\eta^a(k)\Delta t^2 + \sum_{k=i}^{j-1}\delta V_W^\eta(i,k)\Delta t \\
&- \bar{F}(i,j)(\Lambda(\bar{\theta}_B(j))\bar{a}_B(j))^\wedge\delta\phi_{BW}^\eta(i,j)\Delta t^2 \\
&- \frac{1}{3}\bar{F}(i,j)\Lambda(\bar{\theta}_B(j))(\bar{a}_B(j))^\wedge\Phi(-\bar{\theta}_B(j))\eta^g(j)\Delta t^3 \\
&+ \bar{F}(i,j)\Lambda(\bar{\theta}_B(j))\eta^a(j)\Delta t^2 + \delta V_W^\eta(i,j)\Delta t \\
&= \underline{\delta P_W^\eta(i,j)} + \Delta t\,\underline{\delta V_W^\eta(i,j)} \\
&- \bar{F}(i,j)(\Lambda(\bar{\theta}_B(j))\bar{a}_B(j))^\wedge\Delta t^2\,\underline{\delta\phi_{BW}^\eta(i,j)} \\
&- \frac{1}{3}\bar{F}(i,j)\Lambda(\bar{\theta}_B(j))(\bar{a}_B(j))^\wedge\Phi(-\bar{\theta}_B(j))\Delta t^3\,\underline{\eta^g(j)} \\
&+ \bar{F}(i,j)\Lambda(\bar{\theta}_B(j))\Delta t^2\,\underline{\eta^a(j)}
\end{aligned} \tag{24}
$$

4.4 Iterative Noise Form

The preintegrated noise vector can be represented as the following with IMU manufacture specification:

$$\begin{aligned} &\eta_W(i,j) = [(\delta\phi^{\eta}_{BW}(i,j))^T, (\delta V^{\eta}_W(i,j))^T, (\delta P^{\eta}_W(i,j))^T]^T \\ &\eta_{IMU}(k) = [(\eta^g(k))^T, (\eta^a(k))^T]^T \end{aligned} \tag{25}$$

We can represent Equations (22),(23) and (24) the noise propagation in compact matrix form as the following:

Defining a (3×3) matrix $A(j)$ as

$$\begin{cases} A_{11}(j) = \bar{F}^T(j,j+1) \\ A_{12}(j) = A_{13}(j) = 0_{3\times 3} \\ A_{21}(j) = -\bar{F}(i,j)(\Gamma(\bar{\theta}_B(j))\bar{a}_B(j))^{\wedge}\Delta t \\ A_{22}(j) = I_{3\times 3} \\ A_{23}(j) = 0_{3\times 3} \\ A_{31}(j) = -\bar{F}(i,j)(\Lambda(\bar{\theta}_B(j))\bar{a}_B(j))^{\wedge}\Delta t^2 \\ A_{32}(j) = I_{3\times 3}\Delta t \\ A_{33}(j) = I_{3\times 3} \end{cases}, \tag{26}$$

and a (3×2) matrix $B(j)$ as

$$\begin{cases} B_{11}(j) = \Gamma(-\bar{\theta}_B(j))\Delta t \\ B_{12}(j) = 0_{3\times 3} \\ B_{21}(j) = -\frac{1}{2}\bar{F}(i,j)\Gamma(\bar{\theta}_B(j))(\bar{a}_B(j))^{\wedge}\Lambda(-\bar{\theta}_B(j))\Delta t^2 \\ B_{22}(j) = \bar{F}(i,j)\Gamma(\bar{\theta}(j))\Delta t \\ B_{31}(j) = -\frac{1}{3}\bar{F}(i,j)\Lambda(\bar{\theta}_B(j))(\bar{a}_B(j))^{\wedge}\Phi(-\bar{\theta}_B(j))\Delta t^3 \\ B_{32}(j) = \bar{F}(i,j)\Lambda(\bar{\theta}_B(j))\Delta t^2 \end{cases}, \tag{27}$$

it can be derived that

$$\begin{bmatrix} \delta\phi^{\eta}_{BW}(i,j+1) \\ \delta V^{\eta}_W(i,j+1) \\ \delta P^{\eta}_W(i,j+1) \end{bmatrix} = A(j) \begin{bmatrix} \delta\phi^{\eta}_{BW}(i,j) \\ \delta V^{\eta}_W(i,j) \\ \delta P^{\eta}_W(i,j) \end{bmatrix} + B(j) \begin{bmatrix} \eta^g(j) \\ \eta^a(j) \end{bmatrix}. \tag{28}$$

Subsequently, it can be computed the preintegrated measurement covariance as the following:

$$\Sigma(i,j+1) = A(j)\Sigma(i,j)A^T(j) + B(j)\Sigma_{\eta}(j)B^T(j). \tag{29}$$

Where the $\Sigma_{\eta} \in \mathbb{R}^{6\times 6}$ is the raw IMU measurements noise η_{IMU}

5 Bias Update via First-order Approximation

In this subsection, we are trying to find a expression of preintegration measurements update when there is changes in the bias estimation. The main idea to find a small correction plus the old the preintegrated measurements. So, the new bias $\hat{b}(i)$ is equal to the old bias $\bar{b}(i)$ plus small update bias correction $\delta b(i)$.

$$\begin{aligned} \hat{b}^g(i) &= \bar{b}^g(i) + \delta b^g(i) \\ \hat{b}^a(i) &= \bar{b}^a(i) + \delta b^a(i) \end{aligned} \tag{30}$$

By substituting the new value of bias estimation in (30) in preintegration model in (15), we can update the preintegrated measurements.

5.1 Preintegrated Rotational Measurements via Bias Correction

$$\begin{aligned} \Delta R_{BW}(i,j) &\overset{\text{eq.(30)}}{=} \prod_{k=i}^{j-1} E((\tilde{\omega}_B(k) - \bar{b}^g(i) - \delta b^g(i))\Delta t) \\ &\overset{\text{eq.(9)}}{\approx} \prod_{k=i}^{j-1} E((\tilde{\omega}_B(k) - \bar{b}^g(i))\Delta t) E(-\Gamma(-\bar{\theta}_B(k))\delta b^g(i)\Delta t) \\ &\overset{\text{eq.(8)}}{=} \bar{F}(i,j) \prod_{k=i}^{j-1} E(-\bar{F}^T(k+1,j)\Gamma(-\bar{\theta}_B(k))\delta b^g(i)\Delta t) \\ &\overset{\text{eq.(11)}}{\approx} \bar{F}(i,j) E\left(-\sum_{k=i}^{j-1} \bar{F}^T(k+1,j)\Gamma(-\bar{\theta}_B(k))\Delta t \underline{\delta b^g(i)}\right) \\ &= \bar{F}(i,j) E\left(\frac{\partial \bar{F}(i,j)}{\partial b^g} \underline{\delta b^g(i)}\right) \\ &\doteq \bar{F}(i,j) E(\delta \phi^b_{BW}(i,j)) \end{aligned} \tag{31}$$

5.2 Preintegrated Velocity Measurements via Bias Correction

$$
\begin{aligned}
\Delta V_W(i,j) &\overset{\text{eq.(30),(31)}}{=} \sum_{k=i}^{j-1} \bar{F}(i,k) E(\delta\phi_{BW}^b(i,k)) \Gamma(\bar{\theta}_B(k) - \delta b^g(i)\Delta t) \\
&\quad (\bar{a}_B(k) - \delta b^a(i))\Delta t \\
&\overset{\text{eq.(9)}}{\approx} \sum_{k=i}^{j-1} \bar{F}(i,k) E(\delta\phi_{BW}^b(i,k)) \Gamma(\bar{\theta}_B(k)) \Gamma(-\Lambda(-\bar{\theta}_B(k)) \delta b^g(i)\Delta t) \\
&\quad (\bar{a}_B(k) - \delta b^a(i))\Delta t \\
&\overset{\text{eq.(5)}}{\approx} \sum_{k=i}^{j-1} \bar{F}(i,k) (I + \delta\phi_{BW}^b(i,k)^\wedge) \Gamma(\bar{\theta}_B(k)) (I - \frac{1}{2}(\Lambda(-\bar{\theta}_B(k)) \delta b^g(i)\Delta t))^\wedge) \\
&\quad (\bar{a}_B(k) - \delta b^a(i))\Delta t \\
&\overset{\text{eq.(3)}}{\approx} \sum_{k=i}^{j-1} \bar{F}(i,k) \Gamma(\bar{\theta}_B(k)) \bar{a}_B(k) \Delta t \\
&\quad - \sum_{k=i}^{j-1} \bar{F}(i,k) \Gamma(\bar{\theta}_B(k)) \Delta t \underline{\delta b^a(i)} \\
&\quad - \sum_{k=i}^{j-1} \bar{F}(i,k) (\Gamma(\bar{\theta}_B(k)) \bar{a}_B(k))^\wedge \Delta t \underline{\delta\phi_{BW}^b(i,k)} \\
&\quad + \frac{1}{2} \sum_{k=i}^{j-1} \bar{F}(i,k) \Gamma(\bar{\theta}_B(k)) (\bar{a}_B(k))^\wedge \Lambda(-\bar{\theta}_B(k)) \Delta t^2 \underline{\delta b^g(i)} \\
&\doteq \bar{\mu}(i,j) + \frac{\partial \bar{V}(i,j)}{\partial b^a} \delta b^a(i) + \frac{\partial \bar{V}(i,j)}{\partial b^g} \delta b^g(i) \\
&\doteq \bar{\mu}(i,j) + \delta V_W^b(i,j). \qquad (32)
\end{aligned}
$$

5.3 Preintegrated Position Measurements via Bias Correction

$$
\begin{aligned}
\Delta P_W(i,j) &\overset{\text{eq.(30),(32)}}{=} \sum_{k=i}^{j-1}\Big[\bar{F}(i,k)E(\delta\phi_{BW}^b(i,k))\Lambda(\bar{\theta}_B(k)-\delta b^g(i)\Delta t) \\
&\qquad (\bar{a}_B(k)-\delta b^a(i))\Delta t^2+(\bar{\mu}(i,k)+\delta V_W^b(i,k))\Delta t\Big] \\
&\overset{\text{eq.(9)}}{\approx} \sum_{k=i}^{j-1}\Big[\bar{F}(i,k)E(\delta\phi_{BW}^b(i,k))2\Lambda(\bar{\theta}_B(k))\Lambda(-\Phi(-\bar{\theta}_B(k))\delta b^g(i)\Delta t) \\
&\qquad (\bar{a}_B(k)-\delta b^a(i))\Delta t^2+(\bar{\mu}(i,k)+\delta V_W^b(i,k))\Delta t\Big] \\
&\overset{\text{eq.(5)}}{\approx} \sum_{k=i}^{j-1}\Big[\bar{F}(i,k)(I+\delta\phi_{BW}^b(i,k)^\wedge)\Lambda(\bar{\theta}_B(k))(I-\frac{1}{3}(\Phi(-\bar{\theta}_B(k))\delta b^g(i)\Delta t))^\wedge) \\
&\qquad (\bar{a}_B(k)-\delta b^a(i))\Delta t^2+(\bar{\mu}(i,k)+\delta V_W^b(i,k))\Delta t\Big] \\
&\approx \sum_{k=i}^{j-1}\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))\bar{a}_B(k)\Delta t^2+\sum_{k=i}^{j-1}\bar{\mu}(i,k)\Delta t \\
&- \sum_{k=i}^{j-1}\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))\Delta t^2\underline{\delta b^a(i)} \\
&- \sum_{k=i}^{j-1}\bar{F}(i,k)(\Lambda(\bar{\theta}_B(k))\bar{a}_B(k))^\wedge\Delta t^2\underline{\delta\phi_{BW}^b(i,k)} \\
&+ \frac{1}{3}\sum_{k=i}^{j-1}\bar{F}(i,k)\Lambda(\bar{\theta}_B(k))(\bar{a}_B(k))^\wedge\Phi(-\bar{\theta}_B(k))\Delta t^3\underline{\delta b^g(i)} \\
&+ \sum_{k=i}^{j-1}\Delta t\underline{\delta V_W^b(i,k)} \\
&\doteq \bar{\zeta}(i,j)+\frac{\partial\bar{P}(i,j)}{\partial b^a}\delta b^a(i)+\frac{\partial\bar{P}(i,j)}{\partial b^g}\delta b^g(i) \\
&\doteq \bar{\zeta}(i,j)+\delta P_W^b(i,j) \qquad (33)
\end{aligned}
$$

Where the partial derivatives $\frac{\partial \bar{F}(i,j)}{\partial b^g}$, $\frac{\partial \bar{V}(i,j)}{\partial b^g}$, $\frac{\partial \bar{V}(i,j)}{\partial b^a}$, $\frac{\partial \bar{P}(i,j)}{\partial b^g}$ and $\frac{\partial \bar{P}(i,j)}{\partial b^a}$ are given as:

$$
\begin{aligned}
\frac{\partial \bar{F}(i,j)}{\partial b^g} &= -\sum_{k=i}^{j-1} \bar{F}^T(k+1,j)\Gamma(-\bar{\theta}_B(k))\Delta t \\
\frac{\partial \bar{V}(i,j)}{\partial b^g} &= -\sum_{k=i}^{j-1} \bar{F}(i,k)(\Gamma(\bar{\theta}_B(k))\bar{a}_B(k))^\wedge \left(\frac{\partial \bar{F}(i,k)}{\partial b^g}\right)\Delta t \\
&+ \frac{1}{2}\sum_{k=i}^{j-1} \bar{F}(i,k)\Gamma(\bar{\theta}_B(k))(\bar{a}_B(k))^\wedge \Lambda(-\bar{\theta}_B(k))\Delta t^2 \\
\frac{\partial \bar{V}(i,j)}{\partial b^a} &= -\sum_{k=i}^{j-1} \bar{F}(i,k)\Gamma(\bar{\theta}_B(k))\Delta t \\
\frac{\partial \bar{P}(i,j)}{\partial b^g} &= -\sum_{k=i}^{j-1} \bar{F}(i,k)(\Lambda(\bar{\theta}_B(k))\bar{a}_B(k))^\wedge \left(\frac{\partial \bar{F}(i,k)}{\partial b^g}\right)\Delta t^2 \\
&+ \frac{1}{3}\sum_{k=i}^{j-1} \bar{F}(i,k)\Lambda(\bar{\theta}_B(k))(\bar{a}_B(k))^\wedge \Phi(-\bar{\theta}_B(k))\Delta t^3 \\
&+ \sum_{k=i}^{j-1}\left(\frac{\partial \bar{V}(i,k)}{\partial b^g}\right)\Delta t \\
\frac{\partial \bar{P}(i,j)}{\partial b^a} &= -\sum_{k=i}^{j-1} \bar{F}(i,k)\Lambda(\bar{\theta}_B(k))\Delta t^2 + \sum_{k=i}^{j-1}\left(\frac{\partial \bar{V}(i,k)}{\partial b^a}\right)\Delta t
\end{aligned}
\tag{34}
$$

6 Preintegrated Measurement Jacobians

From the preintegrated model (16) and bias update (31)-(33) we can write the residual errors as $r_\Delta = [r^T_{\Delta R(i,j)}, r^T_{\Delta V(i,j)}, r^T_{\Delta P(i,j)}] \in \mathbb{R}^9$, where

$$
\begin{cases}
r_{\Delta R(i,j)} = Log((\bar{F}(i,j)E(\frac{\partial F(i,j)}{\partial b^g(i)}\delta b^g(i)))^T R^T_{BW}(i)R_{BW}(j)) \\
r_{\Delta V(i,j)} = R^T_{BW}(i)(V_W(j) - V_W(i) - g\sum_{k=i}^{j-1}\Delta t) \\
\qquad -(\bar{\mu}(i,j) + \frac{\partial \mu(i,j)}{\partial b^g(i)}\delta b^g(i) + \frac{\partial \mu(i,j)}{\partial b^a(i)}\delta b^a(i)) \\
r_{\Delta P(i,j)} = R^T_{BW}(i)(P_W(j) - P_W(i) - \Theta(i,j)) \\
\qquad -(\bar{\zeta}(i,j) + \frac{\partial \zeta(i,j)}{\partial b^g(i)}\delta b^g(i) + \frac{\partial \zeta(i,j)}{\partial b^a(i)}\delta b^a(i))
\end{cases}
. \tag{35}
$$

"Lifting" the cost function consists of substituting the following retractions:

$$\begin{cases} R_{BW}(i) \leftarrow R_{BW}(i)E(\delta\phi(i)) \\ R_{BW}(j) \leftarrow R_{BW}(j)E(\delta\phi(j)) \\ P_W(i) \leftarrow P_W(i) + R_{BW}(i)\delta P(i) \\ P_W(j) \leftarrow P_W(j) + R_{BW}(j)\delta P(j) \\ V_W(i) \leftarrow V_W(i) + \delta V(i) \\ V_W(j) \leftarrow V_W(j) + \delta V(j) \\ \delta b^g(i) \leftarrow \delta b^g(i) + \tilde{\delta} b^g(i) \\ \delta b^a(i) \leftarrow \delta b^a(i) + \tilde{\delta} b^a(i) \end{cases} . \tag{36}$$

6.1 Jacobians of Position Residual Error ($r_{\Delta P(i,j)}$)

$$\begin{aligned} r_{\Delta P(i,j)} = & R_{BW}^T(i)(P_W(j) - P_W(i) - \Theta(i,j)) \\ & -\bar{\zeta}(i,j) - \frac{\partial \zeta(i,j)}{\partial b^g(i)}\delta b^g(i) - \frac{\partial \zeta(i,j)}{\partial b^a(i)}\delta b^a(i) \end{aligned} \tag{37}$$

Both $R_{BW}(j)$ and $V_W(j)$ do not exist in $r_{\Delta P(i,j)}$ in (37), accordingly the jacobians w.r.t $\delta\phi(j)$ and $\delta V(j)$ are zero. On the other hand, $r_{\Delta P(i,j)}$ is a linear function in both $\delta b^g(i)$ and $\delta b^a(i)$, hence the jacobians w.r.t $\tilde{\delta} b^g(i)$ and $\tilde{\delta} b^a(i)$ are the same coefficients of $\delta b^g(i)$ and $\delta b^a(i)$ in (34).

For jacobians w.r.t $\delta P(i)$

$$\begin{aligned} r_{\Delta P(i,j)}(P_W(i) + R_{BW}(i)\delta P(i)) &= R_{BW}^T(i)(P_W(j) - P_W(i) - R_{BW}(i)\delta P(i) - \Theta(i,j)) - C \\ &= r_{\Delta P(i,j)}(P_W(i)) - I_{3\times 1}\delta P(i) \end{aligned} \tag{38}$$

Where $C \doteq \bar{\zeta}(i,j) + \frac{\partial \zeta(i,j)}{\partial b^g(i)}\delta b^g(i) + \frac{\partial \zeta(i,j)}{\partial b^a(i)}\delta b^a(i)$

For jacobians w.r.t $\delta P(j)$

$$\begin{aligned} r_{\Delta P(i,j)}(P_W(j) + R_{BW}(j)\delta P(j)) &= R_{BW}^T(i)(P_W(j) + R_{BW}(j)\delta P(j) - P_W(i) - \Theta(i,j)) - C \\ &= r_{\Delta P(i,j)}(P_W(j)) - R_{BW}^T(i)R_{BW}(j)\delta P(j) \end{aligned} \tag{39}$$

For jacobians w.r.t $\delta V(i)$

$$
\begin{aligned}
r_{\Delta P(i,j)}(V_W(i)+\delta V(i)) &= R_{BW}^T(i)\left(P_W(j)-P_W(i)-V_W(i)\sum_{k=i}^{j-1}\Delta t-\delta V(i)\sum_{k=i}^{j-1}\Delta t\right. \\
&- \left.\frac{g_W}{2}(\sum_{k=i}^{j-1}\Delta t)^2\right)-C \\
&= r_{\Delta P(i,j)}(V_W(i))-R_{BW}^T(i)\sum_{k=i}^{j-1}\Delta t\,\delta V(i)
\end{aligned} \tag{40}
$$

For jacobians w.r.t $\delta\phi(i)$

$$
\begin{aligned}
r_{\Delta P(i,j)}(R_{BW}(i)E(\delta\phi(i))) &= (R_{BW}(i)E(\delta\phi(i))^T(P_W(j)-P_W(i)-\Theta(i,j))-C \\
&\overset{\text{eq.(5)}}{\approx} (I-\delta\phi(i)^\wedge)R_{BW}^T(i)(P_W(j)-P_W(i)-\Theta(i,j))-C \\
&\overset{\text{eq.(3)}}{=} r_{\Delta P(i,j)}(R_{BW}(i))+\left(R_{BW}^T(i)(P_W(j)-P_W(i)-\Theta(i,j))\right)^\wedge\delta\phi(i)
\end{aligned} \tag{41}
$$

the Jacobians of $r_{\Delta P(i,j)}$ are computed as

$$
\begin{cases}
\frac{\partial r_{\Delta P(i,j)}}{\partial\delta\phi(i)} = (R_{BW}^T(i)(P_W(j)-P_W(i)-\Theta(i,j)))^\wedge \\
\frac{\partial r_{\Delta P(i,j)}}{\partial\delta\phi(j)} = 0_{3\times1} \\
\frac{\partial r_{\Delta P(i,j)}}{\partial\delta P(i)} = -I_{3\times1} \\
\frac{\partial r_{\Delta P(i,j)}}{\partial\delta P(j)} = R_{BW}^T(i)R_{BW}(j) \\
\frac{\partial r_{\Delta P(i,j)}}{\partial\delta V(i)} = -R_{BW}^T(i)\sum_{k=i}^{j-1}\Delta t \\
\frac{\partial r_{\Delta P(i,j)}}{\partial\delta V(j)} = 0_{3\times1} \\
\frac{\partial r_{\Delta P(i,j)}}{\partial\tilde{\delta}b^a(i)} = -\frac{\partial\bar{P}(i,j)}{\partial b^a} \\
\frac{\partial r_{\Delta P(i,j)}}{\partial\tilde{\delta}b^g(i)} = -\frac{\partial\bar{P}(i,j)}{\partial b^g}
\end{cases}, \tag{42}
$$

6.2 Jacobians of Velocity Residual Error ($r_{\Delta V(i,j)}$)

$$\begin{aligned} r_{\Delta V(i,j)} = R_{BW}^T(i)(V_W(j) - V_W(i) - g\sum_{k=i}^{j-1}\Delta t) \\ -\bar{\mu}(i,j) - \tfrac{\partial \mu(i,j)}{\partial b^g(i)}\delta b^g(i) - \tfrac{\partial \mu(i,j)}{\partial b^a(i)}\delta b^a(i) \end{aligned} \tag{43}$$

Both $R_{BW}(j)$, $P_W(j)$ and $P_W(i)$ do not exist in $r_{\Delta V(i,j)}$ in (43), accordingly the jacobians w.r.t $\delta\phi(j)$, $\delta p(j)$ and $\delta p(i)$ are zero. On the other hand, $r_{\Delta V(i,j)}$ is a linear function in both $\delta b^g(i)$ and $\delta b^a(i)$, hence the jacobians w.r.t $\tilde{\delta} b^g(i)$ and $\tilde{\delta} b^a(i)$ are the same coefficients of $\delta b^g(i)$ and $\delta b^a(i)$ in (34).

For jacobians w.r.t $\delta V(i)$

$$\begin{aligned} r_{\Delta V(i,j)}(V_W(i) + \delta V(i)) &= R_{BW}^T(i)(V_W(j) - V_W(i) - \delta V(i) - g\sum_{k=i}^{j-1}\Delta t) - D \\ &= r_{\Delta V(i,j)}(V_W(i)) - R_{BW}^T(i)\delta V(i) \end{aligned} \tag{44}$$

Where $D \doteq \bar{\mu}(i,j) + \frac{\partial \mu(i,j)}{\partial b^g(i)}\delta b^g(i) + \frac{\partial \mu(i,j)}{\partial b^a(i)}\delta b^a(i)$

For jacobians w.r.t $\delta V(j)$

$$\begin{aligned} r_{\Delta V(i,j)}(V_W(j) + \delta V(j)) &= R_{BW}^T(i)(V_W(j) + \delta V(j) - V_W(i) - g\sum_{k=i}^{j-1}\Delta t) - D \\ &= r_{\Delta V(i,j)}(V_W(j)) + R_{BW}^T(i)\delta V(j) \end{aligned} \tag{45}$$

For jacobians w.r.t $\delta\phi(i)$

$$\begin{aligned} r_{\Delta V(i,j)}(R_{BW}(i)E(\delta\phi(i))) &= (R_{BW}(i)E(\delta\phi(i))^T(V_W(j) - V_W(i) - g\sum_{k=i}^{j-1}\Delta t) - D \\ &\overset{\text{eq.(5)}}{\approx} (I - \delta\phi(i)^\wedge)R_{BW}^T(i)(V_W(j) - V_W(i) - g\sum_{k=i}^{j-1}\Delta t) - D \\ &\overset{\text{eq.(3)}}{=} r_{\Delta V(i,j)}(R_{BW}(i)) + \left(R_{BW}^T(i)(V_W(j) - V_W(i) - g\sum_{k=i}^{j-1}\Delta t)\right)^\wedge \delta\phi(i) \end{aligned} \tag{46}$$

the Jacobians of $r_{\Delta V(i,j)}$ are computed as

$$\begin{cases} \frac{\partial r_{\Delta V(i,j)}}{\partial \delta\phi(i)} = (R_{BW}^T(i)(V_W(j) - V_W(i) - g\sum_{k=i}^{j-1}\Delta t))^{\wedge} \\ \frac{\partial r_{\Delta V(i,j)}}{\partial \delta\phi(j)} = 0_{3\times 1} \\ \frac{\partial r_{\Delta V(i,j)}}{\partial \delta P(i)} = 0_{3\times 1} \\ \frac{\partial r_{\Delta V(i,j)}}{\partial \delta P(j)} = 0_{3\times 1} \\ \frac{\partial r_{\Delta V(i,j)}}{\partial \delta V(i)} = -R_{BW}^T(i) \\ \frac{\partial r_{\Delta V(i,j)}}{\partial \delta V(j)} = R_{BW}^T(i) \\ \frac{\partial r_{\Delta V(i,j)}}{\partial \tilde{\delta} b^a(i)} = -\frac{\partial \bar{V}(i,j)}{\partial b^a} \\ \frac{\partial r_{\Delta V(i,j)}}{\partial \tilde{\delta} b^g(i)} = -\frac{\partial \bar{V}(i,j)}{\partial b^g} \end{cases}, \quad (47)$$

6.3 Jacobians of Rotational Residual Error ($r_{\Delta R(i,j)}$)

$$r_{\Delta R(i,j)} = Log((\bar{F}(i,j)E(\frac{\partial F(i,j)}{\partial b^g(i)}\delta b^g(i)))^T R_{BW}^T(i)R_{BW}(j)) \quad (48)$$

Both of $P_W(i)$, $P_W(j)$, $V_W(i)$, $V_W(j)$ and $\delta b^a(i)$ do not exist in $r_{\Delta R(i,j)}$ in (48), accordingly the jacobians w.r.t $\delta p(i)$, $\delta p(j)$, $\delta V(i)$, $\delta V(j)$ and $\tilde{\delta} b^a(i)$ are zero.

For jacobians w.r.t $\delta\phi(i)$

$$\begin{aligned} r_{\Delta R(i,j)}(R_{BW}(i)E(\delta\phi(i))) &= Log((\bar{F}(i,j)U)^T(R_{BW}(i)E(\delta\phi(i)))^T R_{BW}(j)) \\ &= Log((\bar{F}(i,j)U)^T E(-\delta\phi(i))R_{BW}^T(i)R_{BW}(j)) \\ &\overset{\text{eq.(8)}}{=} Log((\bar{F}(i,j)U)^T R_{BW}^T(i)R_{BW}(j)E(-R_{BW}^T(j)R_{BW}(i)\delta\phi(i))) \\ &\overset{\text{eq.(11)}}{\approx} r_{\Delta R(i,j)} - \Gamma^{-1}(-r_{\Delta R(i,j)})R_{BW}^T(j)R_{BW}(i)\delta\phi(i) \end{aligned} \quad (49)$$

Where $U \doteq E(\frac{\partial F(i,j)}{\partial b^g(i)}\delta b^g(i))$

For jacobians w.r.t $\delta\phi(j)$

$$\begin{aligned} r_{\Delta R(i,j)}(R_{BW}(j)E(\delta\phi(j))) &= Log((\bar{F}(i,j)U)^T(R_{BW}^T(i)R_{BW}(j)E(\delta\phi(j))) \\ &\overset{\text{eq.(11)}}{\approx} r_{\Delta R(i,j)} + \Gamma^{-1}(-r_{\Delta R(i,j)})\delta\phi(i) \end{aligned} \quad (50)$$

For jacobians w.r.t $\tilde{\delta} b^g(i)$

$$
\begin{aligned}
r_{\Delta R(i,j)}(\delta b^g(i)+\tilde{\delta} b^g(i)) &= Log\left(\left(\bar{F}(i,j)E\left(\frac{\partial F(i,j)}{\partial b^g(i)}(\delta b^g(i)+\tilde{\delta} b^g(i))\right)\right)^T R_{BW}^T(i)R_{BW}(j)\right)\\
&\overset{\text{eq.(9)}}{\approx} Log\left(\left(\bar{F}(i,j)UE\left(\Gamma(-M)\frac{\partial F(i,j)}{\partial b^g(i)}\tilde{\delta} b^g(i)\right)\right)^T R_{BW}^T(i)R_{BW}(j)\right)\\
&= Log\left(E\left(\Gamma(M)\frac{\partial F(i,j)}{\partial b^g(i)}\tilde{\delta} b^g(i)\right)(\bar{F}(i,j)U)^T R_{BW}^T(i)R_{BW}(j)\right)\\
&\overset{\text{eq.(8)}}{=} Log\left(E(r_{\Delta R(i,j)})E\left(E\left(r_{\Delta R(i,j)}\right)^T\left(\Gamma(M)\frac{\partial F(i,j)}{\partial b^g(i)}\tilde{\delta} b^g(i)\right)\right)\right)\\
&\overset{\text{eq.(11)}}{=} r_{\Delta R(i,j)}-\Gamma^{-1}(-r_{\Delta R(i,j)})E\left(r_{\Delta R(i,j)}\right)^T\left(\Gamma(M)\frac{\partial F(i,j)}{\partial b^g(i)}\tilde{\delta} b^g(i)\right)
\end{aligned}
\tag{51}
$$

Where $M \doteq \frac{\partial F(i,j)}{\partial b^g(i)}\delta b^g(i)$

and the Jacobians of $r_{\Delta R(i,j)}$ are computed as

$$
\begin{cases}
\frac{\partial r_{\Delta R(i,j)}}{\partial \delta\phi(i)} = -\Gamma^{-1}(-r_{\Delta R(i,j)})R_{BW}^T(j)R_{BW}(i)\\
\frac{\partial r_{\Delta R(i,j)}}{\partial \delta\phi(j)} = \Gamma^{-1}(-r_{\Delta R(i,j)})\\
\frac{\partial r_{\Delta R(i,j)}}{\partial \delta P(i)} = 0_{3\times 1}\\
\frac{\partial r_{\Delta R(i,j)}}{\partial \delta P(j)} = 0_{3\times 1}\\
\frac{\partial r_{\Delta R(i,j)}}{\partial \delta V(i)} = 0_{3\times 1}\\
\frac{\partial r_{\Delta R(i,j)}}{\partial \delta V(j)} = 0_{3\times 1}\\
\frac{\partial r_{\Delta R(i,j)}}{\partial \tilde{\delta} b^a(i)} = 0_{3\times 1}\\
\frac{\partial r_{\Delta R(i,j)}}{\partial \tilde{\delta} b^g(i)} = -\Gamma^{-1}(-r_{\Delta R(i,j)})E^T(r_{\Delta R(i,j)})\Gamma(M)\frac{\partial F(i,j)}{\partial b^g(i)}
\end{cases}
. \tag{52}
$$

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