

Energy efficient network-flow-based algorithm for multiuser multicarrier systems

J. Joun¹ D. Yuan² C. K. Ho¹ S. Sun¹

¹Institute for Infocomm Research (I²R), A*STAR, Singapore

E-mail: {jjjoun, hock, sunsm}@i2r.a-star.edu.sg

²Department of Science and Technology, Linköping University, Sweden

E-mail: diyua@itn.liu.se

Abstract: The resource allocation problem of minimizing transmission power with per user rate constraint is studied for energy efficiency of multiuser multicarrier systems. We introduce an algorithm that deploys a flow-based decomposition strategy, called the network-flow-based algorithm (NFBA), to circumvent the NP-hard complexity of the resource allocation problem. We prove global optimality of the algorithm for the case of a flat-fading channel. For the general frequency selective channel, the flow size is adjusted adaptively to minimize the transmission power. Also, a compact integer programming formulation is developed to provide tight lower and upper bounds of the minimum power. Numerical results show that our proposed NFBA achieves near-optimal performance with polynomial complexity even for large-scale systems.

1 Introduction

Resource allocation including subchannel (alternatively subcarrier or subband) allocation and power assignment has been actively considered in multiuser multicarrier systems, such as in orthogonal frequency-division multiple access (OFDMA) [1, 2]. Due to the increasing concern of energy efficiency, recently much focus for resource allocation has been on minimizing energy consumption. The energy optimization applies to most types of wireless networks, such as cellular networks which is the main application area of our work, and ad hoc networks (e.g., [3]). The resource allocation problem for energy minimization is, however, NP-hard [4, 5]. Variable relaxation, asymptotical approaches, and heuristic algorithms have been applied to circumvent the extremely high complexity to obtain near optimal subchannel allocations [6, 7, 8, 9, 10]. For example, minimization of weighted sum power and maximization of sum or minimum rate have been studied [6, 7, 8]. In [9], the bandwidth-assignment-based-on-SNR (BABS) algorithm decides the number of subcarriers each user gets, and the rate-craving-greedy (RCG) algorithm or the amplitude-craving-greedy (ACG) algorithm is then used to assign specific subcarriers and power allocation to users. The RCG and ACG algorithms are greedy approaches that assign subcarriers based on the maximum rate and the maximum effective channel, respectively. In [10], the successive user integration (SUSI) algorithm is proposed which proceeds in two stages. After selecting an initial candidate, the SUSI algorithm performs a local search algorithm, i.e., it searches around neighboring solution candidates, by continuously exploring if there is a net decrease in the transmission power by re-assigning one subchannel from user to another user. The algorithm stops when it reaches a local optimum, at which none of the neighboring solutions leads to any improvement. Many other two-stage algorithms have been proposed recently, including sorting-and-assignment algorithms [11, 14], subcarrier-and-power allocation algorithms [12], and user-and-rate allocation algorithms [13]. The aforementioned heuristic algorithms, however, achieve loose gaps from the optimum, or their complexities are hard to predict as the number of iterations needed is a tuning parameter. Hence, it is of practical importance to study simple strategies that achieve near-optimal performance with tractable complexity.

In this paper, we focus on developing an adaptive heuristic algorithm with near-optimal performance and polynomial complexity within a general framework. The resulting algorithm is referred to as the network-flow-based algorithm (NFBA). The proposed NFBA sequentially allocates F^t resources in the t th iteration, until all resources are exhausted. A graph-based solution framework (i.e., minimum cost multi-commodity network flow) has been performed for utility maximization of OFDMA resource allocation [15]. The utility is maximized by minimizing the flow cost representing the negative values of *data rate*. In contrast to maximum-utility resource allocation, the problem of minimum power subject to rate target that we consider does not admit a single-stage multi-commodity flow formulation. In the proposed NFBA, we maximize the *potential power saving* (profit) on the flow, instead of minimizing the cost, in a sequence of stages. We study the key problem of deciding F^t adaptively. To this end, we first consider NFBA-1 and NFBA-max in which the flow size is fixed as its minimum ($F^t = 1$) and maximum ($F^t = \max(N^t, M)$), respectively, where N^t is the number of subchannels available in iteration t and M is the number of users. Moreover, we justify the rationale of NFBA by proving that NFBA-1 guarantees global optimality for flat fading channels. To achieve both merits of NFBA-1 and NFBA-max, the flow size F^t is adapted according to the frequency selectivity of the channel, resulting in the so-called NFBA-adapt algorithm. For the adaptation, we employ a uniformity metric that reflects the power saving variation (from channel variation). We show that the complexity of the NFBA with adaptation is polynomial. We also develop a compact integer programming formulation based on bounding the rate function to provide tight lower and upper bounds of the minimum power. In summary, our contributions include:

- A general NFBA representation for an orthogonal subchannel allocation problem.
- Complexity analysis of NFBA-1, NFBA-max, and NFBA-adapt.
- Optimality guarantee of NFBA-1 in flat-fading channel and of NFBA-max when $M = N$.
- A compact integer programming formulation providing tight lower and upper bounds of the minimum power.

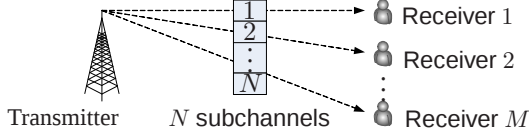


Fig. 1: Illustration of OFDMA downlink communications: transmission from one transmitter to M receivers through N subchannels.

Numerical results show that the bounds obtained from the integer programming problem are within 0.35%, and NFBA-adapt achieves near-optimal performance to within 1.98% of the minimum energy for 20 users with 50 subchannels.

2 Problem formulation

Consider a multiuser multicarrier system in which one transmitter communicates with M receivers via N subchannels as shown in Fig. 1. Denote $\mathcal{M} = \{1, \dots, M\}$ and $\mathcal{N} = \{1, \dots, N\}$ by the sets of M users and N subchannels, respectively. In the OFDMA system, our task is to allocate each user $m \in \mathcal{M}$ to a subchannel set $\mathcal{N}_m \subseteq \mathcal{N}$. Also, we are to allocate the transmission power $P_{mn} \geq 0$ to user m on subchannel n , which results in rate $r_{mn}(P_{mn}) = \log_2(1 + P_{mn}g_{mn})$, where $g_{mn} > 0$ is the channel gain. Here, without loss of generality, we assume a unit variance noise. Hence the signal-to-noise ratio (SNR) is $P_{mn}g_{mn}$. Each user $m \in \mathcal{M}$ has its own target rate R_m , i.e., $\sum_{n \in \mathcal{N}_m} r_{mn}(P_{mn}) \geq R_m$. It is well-known that for each user $m \in \mathcal{M}$, given subchannel allocation $\mathcal{N}_m$ and target rate R_m , the minimum energy (or interchangeably power) allocation for any channel $n \in \mathcal{N}_m$ is unique and given by the so-called water-filling solution [10], denoted by $P_{mn}^*(\mathcal{N}_m)$.

In this paper, we jointly optimize the subchannel allocation and power allocation to minimize the total energy. Since the optimal power allocation is $P_{mn}^*(\mathcal{N}_m)$, the essence of resource allocation is to find the optimal subchannel sets $\{\mathcal{N}_1, \dots, \mathcal{N}_M\}$. Mathematically, our original optimization problem is formulated as:

$$P_{\text{orig}} : \underset{\{\mathcal{N}_m \subseteq \mathcal{N}, \forall m\}}{\text{minimize}} \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}_m} P_{mn}^*(\mathcal{N}_m) \quad (1a)$$

$$\text{s.t. } \mathcal{N}_{m_1} \cap \mathcal{N}_{m_2} = \emptyset, \forall m_1 \neq m_2 \in \mathcal{M} \quad (1b)$$

where the constraint is for orthogonal (exclusive) subchannel allocation among users. Because the orthogonal subchannel allocation in OFDMA systems is an optimal strategy to maximize sum rate as proven in [16], we retain it in our power minimization problem to preserve backward compatibility for the OFDMA systems.

Instead of optimally solving the original problem P_{orig} which is NP hard even if the cardinalities of $\{\mathcal{N}_1, \dots, \mathcal{N}_M\}$ are given [5], we propose a network-flow-based algorithm (NFBA) based on graph-based approach and aimed at near-optimal solutions with tractable polynomial complexity.

3 Network-flow-based algorithm

Before introducing NFBA, we reformulate the original problem P_{orig} in (1) as P'_{orig} in Subsection 3.1. In Subsection 3.2, NFBA

will be described after recapitulating the maximum flow problem with weighted arcs [18], and its flow size adaptation will be proposed in Subsection 3.3.

3.1 Problem reformulation

Let t be an iteration index in iteration set $\mathcal{T} = \{1, \dots, T\}$, where T is the iteration number which will be decided later. Consider the following optimization problem:

$$P'_{\text{orig}} : \underset{\{\mathcal{N}^t, f_{mn}^t, \forall m, n, t\}}{\text{maximize}} \sum_{m \in \mathcal{M}} \sum_{t \in \mathcal{T}} \sum_{n \in \mathcal{N}^t} \delta_{mn}^t f_{mn}^t \quad (2a)$$

$$\text{s.t. } f_{mn}^t \in \{0, 1\}, \forall m, n, t \quad (2b)$$

$$\sum_{n \in \mathcal{N}^t} f_{mn}^t \leq 1, \forall m, t \quad (2c)$$

$$\sum_{m \in \mathcal{M}} \sum_{t \in \mathcal{T}} \sum_{n \in \mathcal{N}^t} f_{mn}^t \leq N \quad (2d)$$

where $\mathcal{N}^t$ is the remaining subchannel set at iteration t ; f_{mn}^t is an assignment indicator that is one if subchannel n is assigned to user m at iteration t , and zero otherwise; and δ_{mn}^t is the potential power saving (profit) for user m at iteration t . If we allocate subchannel n to user m at iteration t , the profit is written as

$$\delta_{mn}^t \triangleq \sum_{n' \in \mathcal{N}_m^{t-1}} P_{mn'}^*(\mathcal{N}_m^{t-1}) - \sum_{n' \in \mathcal{N}_m^{t-1} \cup \{n\}} P_{mn'}^*(\mathcal{N}_m^{t-1} \cup \{n\}) \quad (3)$$

where $\mathcal{N}_m^t$ is the subchannels allocated to user m until iteration t with $\mathcal{N}_m^0 = \emptyset, \forall m \in \mathcal{M}$. We can show that P'_{orig} is equivalent to P_{orig} , as follows. First, note that the constraints (2b) and (2c) ensure orthogonality of all subchannel allocated for all user $m \in \mathcal{M}$ and iteration $t \in \mathcal{T}$, and hence they are equivalent to the constraint in (1). The last constraint (2d) is for the iteration number. From (2a), moreover, we can show that

$$\begin{aligned} \sum_{t \in \mathcal{T}} \sum_{n \in \mathcal{N}^t} \delta_{mn}^t f_{mn}^t &= \\ &= \sum_{n' \in \mathcal{N}_m^{T-1}} P_{mn'}^*(\mathcal{N}_m^{T-1}) - \sum_{n' \in \mathcal{N}_m^{T-1} \cup \{n^T\}} P_{mn'}^*(\mathcal{N}_m^{T-1} \cup \{n^T\}) \\ &+ \sum_{n' \in \mathcal{N}_m^{T-2}} P_{mn'}^*(\mathcal{N}_m^{T-2}) - \sum_{n' \in \mathcal{N}_m^{T-2} \cup \{n^{T-1}\}} P_{mn'}^*(\mathcal{N}_m^{T-2} \cup \{n^{T-1}\}) \\ &+ \dots - \dots \\ &+ \sum_{n' \in \emptyset} P_{mn'}^*(\emptyset) - \sum_{n' \in \emptyset \cup \{n^1\}} P_{mn'}^*(\emptyset \cup \{n^1\}) \\ &= \sum_{n' \in \emptyset} P_{mn'}^*(\emptyset) - \sum_{n' \in \mathcal{N}_m^T} P_{mn'}^*(\mathcal{N}_m^T) \end{aligned} \quad (4)$$

where for all $t \in \{1, \dots, T\}$, $\mathcal{N}_m^t = \mathcal{N}_m^{t-1} \cup \{n^t\}$ and n^t represents the assigned subchannel index at t (i.e., $f_{mn}^t = 0$, if $n \neq n^t$ for all t). Here, $\sum_{n' \in \emptyset} P_{mn'}^*(\emptyset) = P_m^*(\emptyset)$ is an initial power value of user m , which can be set as a constant (i.e., independent from m) that is sufficiently large such that at least one subchannel is allocated to every user. For example, $P_m^*(\emptyset)$ can be set to the power satisfying the maximum rate, i.e., $\max_{m \in \mathcal{M}} R_m$, on the subchannel assuming the minimum gain,

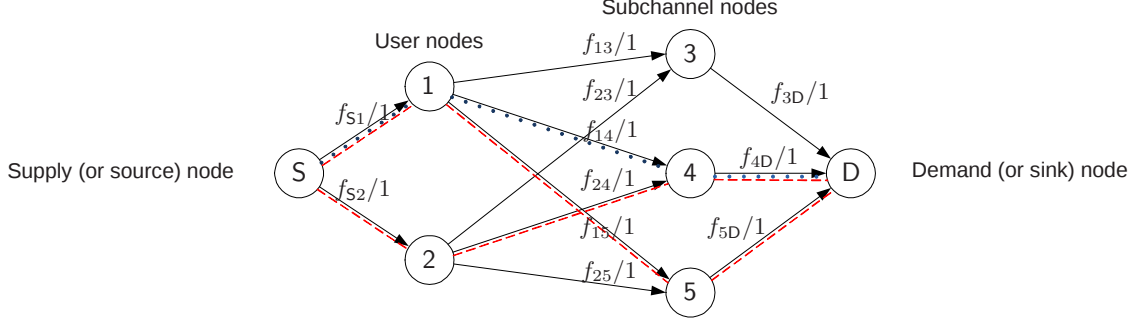


Fig. 2: Example of a network with $M = 2$ and $N^t = 3$. Circle and solid line with arrow represent node and arc with (flow $f_{v_1 v_2}$ /capacity 1), respectively. The dotted line represents an example of allocation snapshot when $F^t = 1$, in which the second subchannel ($n = 2$, denoted by node $\tilde{n} = 4$) is allocated to user 1. The dashed lines represent an example of allocation snapshot when $F^t = 2$, in which the second subchannel is allocated to user 2 and the third subchannel ($n = 3$, denoted by node $\tilde{n} = 5$) is allocated to user 1.

i.e., $\min_{m \in \mathcal{M}, n \in \mathcal{N}} g_{mn}$. Consequently, maximizing the profit in (2a) is identical to minimizing the cost in (1).

The reformulated problem (2) is still intractable because all $\mathcal{N}^t$'s should be entirely optimized for all t , (equivalently $\mathcal{N}_m, \forall m$ in (1) or $\mathcal{N}_m^T, \forall m$ in (4)). Nevertheless, the formulation in (2) allows us to devise NFBA to simplify the original problem.

3.2 NFBA description

To tackle the complexity of P'_{orig} (or equivalently P_{orig}), NFBA sequentially allocates F^t subchannels in iteration t , with $1 \leq F^t \leq M$, until $\sum_t F^t = N$. In other words, for given $\{\mathcal{N}_m^{t-1}, \forall m\}$ and remaining subchannel set $\mathcal{N}^t$ at iteration t , we formulate the sub-problem of P'_{orig} as

$$P_{\text{orig}}^t : \text{maximize}_{\{f_{mn}^t, \forall m, n\}} \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}^t} \delta_{mn}^t f_{mn}^t \quad (5a)$$

$$\text{s.t. } f_{mn}^t \in \{0, 1\}, \forall m, n \quad (5b)$$

$$\sum_{n \in \mathcal{N}^t} f_{mn}^t \leq 1, \forall m \quad (5c)$$

$$\sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}^t} f_{mn}^t = F^t \quad (5d)$$

and solve it from $t = 1$ to $t = T$ sequentially, which is a suboptimal approach to solve P'_{orig} . During the iterations, previously allocated resources are not re-allocated subsequently, hence clearly $F^t \leq \min(N^t, M)$, and we can reduce the computational complexity. We wish to allocate F^t subchannels with the restriction that each user in $\mathcal{M}$ is allocated at most one subchannel at iteration t .

To solve the sub-problem P_{orig}^t via maximum weighted flow, we first recapitulate the maximum flow problem. Consider a network represented by a directed graph. Every arc from vertex v_1 to v_2 ($v_1 \neq v_2$) can deliver a flow of $f_{v_1 v_2} \leq c_{v_1 v_2}$ to give a weighted flow of $\delta_{v_1 v_2} f_{v_1 v_2}$, where the capacity $c_{v_1 v_2} \geq 0$ and the weights $\delta_{v_1 v_2} \geq 0$ on arcs are fixed. The flow starts from one node (supply or source) and ends at another node (demand or sink). The optimization task is to find a flow pattern to maximize the total weighted flow $\sum_{v_1} \sum_{v_2} \delta_{v_1 v_2} f_{v_1 v_2}$, subject to the flow conservation at the network nodes v , i.e., $\sum_{v_1} f_{v_1 v} = \sum_{v_2} f_{v v_2}$.

Now, we reformulate P_{orig}^t as the maximum flow problem with weighted arcs. We construct a network with one source, M user nodes, N^t subchannel nodes and one sink; see example in Fig. 2 with $M = 2$ and $N^t = 3$. The profit δ_{mn}^t and the assignment indicator f_{mn}^t in (5) can be interpreted as the weight and flow, respectively, in the network. To make the node indices unique, we map each subchannel $n \in \mathcal{N}$ to the subchannel node $\tilde{n} \triangleq n + M$. The supply node S is connected to each user node m only; each user node is connected to every subchannel node $\tilde{n}$ only; each subchannel node is connected to the demand node D only. All arcs have unit capacity. The arc from user node m to subchannel node $\tilde{n}$ (i.e., to subchannel n) has weight $\delta_{m\tilde{n}}^t$ and the remaining arcs connected to S or D have zero weight. Then, we can equivalently reformulate P_{orig}^t as follows:

$$P_{\text{flow}}^t : \text{maximize}_{\{f_{m\tilde{n}}^t, f_{Sm}^t, f_{\tilde{n}D}^t, \forall m, \tilde{n}\}} \sum_{m \in \mathcal{M}} \sum_{\tilde{n} \in \mathcal{R}^t} \delta_{m\tilde{n}}^t f_{m\tilde{n}}^t \quad (6a)$$

$$\text{s.t. } f_{m\tilde{n}}^t, f_{Sm}^t, f_{\tilde{n}D}^t \in \{0, 1\}, \forall m, \tilde{n} \quad (6b)$$

$$\sum_{m \in \mathcal{M}} f_{Sm}^t = F^t \quad (6c)$$

$$\sum_{\tilde{n} \in \mathcal{R}^t} f_{\tilde{n}D}^t = F^t \quad (6d)$$

$$\sum_{\tilde{n} \in \mathcal{R}^t} f_{m\tilde{n}}^t = f_{Sm}^t, \forall m \quad (6e)$$

$$\sum_{m \in \mathcal{M}} f_{m\tilde{n}}^t = f_{\tilde{n}D}^t, \forall \tilde{n} \quad (6f)$$

Description of the variables, objective function, and constraints:

Variables:

- flow $f_{m\tilde{n}}^t, f_{Sm}^t, f_{\tilde{n}D}^t, \forall m, \tilde{n}$ are flow variables.

Objective function:

- The objective function in (6a) is a total weighted flow; $\sum_{\tilde{n} \in \mathcal{R}^t} \delta_{m\tilde{n}}^t f_{m\tilde{n}}^t$ is equivalent to δ_m^t in (5a); and $\mathcal{R}^t$ is a remaining subchannel node set just before iteration t with $\mathcal{R}^1 = \{1 + M, \dots, N + M\}$.

Constraints:

- (6b) is a capacity constraint.

- (6c) and (6d) formulate the production (outgoing) and consumption (incoming) of flow at the source and sink nodes, respectively.
- (6e) and (6f) model flow conservation with zero net flow at user and subchannel nodes, respectively.

Assignment mapping:

- $\mathcal{N}_m \leftarrow \bigcup_{\forall t, \tilde{n}} \{n | n = \tilde{n} - M \text{ where } f_{m\tilde{n}}^t = 1\}, \forall m.$

Note that constraints in (6b)–(6f) sustain the constraints in (5b) and (5c). We can readily solve P_{flow}^t through a network flow algorithm or solver. Because of the total unimodularity of minimum-cost flow [18] and unit capacity, the optimum flows $\{f_{m\tilde{n}}^*\}$ on all arcs are either zero or one, even if the integrality requirement (6b) are replaced by the linear constraints $0 \leq f_{m\tilde{n}} \leq 1$, $0 \leq f_{s\tilde{n}} \leq 1$, and $0 \leq f_{\tilde{n}D} \leq 1$. The assignment with F^t is repeated until all subchannels are allocated, i.e., $\mathcal{R}^t = \emptyset$.

A pseudo-code of NFBA is given in Algorithm 1.

Algorithm 1: Network-Flow-Based Algorithm (NFBA)
 set up $t=1$, $\mathcal{R}^1 = \{\tilde{n} : \tilde{n} = n+M; n \in \mathcal{N}\}$, and $\mathcal{N}_m^0 = \emptyset, \forall m$.
while $\mathcal{R}^t \neq \emptyset$
 compute $\delta_{m\tilde{n}}^t$ in (3), $\forall m \in \mathcal{M}$ and $\forall \tilde{n} \in \mathcal{R}^t$.
 find $\{f_{m\tilde{n}}^t\}$ by solving (6a)–(6f) for a given F^t .
 update $\mathcal{N}_m^t = \mathcal{N}_m^{t-1} \cup \{n\}$ and $\mathcal{R}^{t+1} = \mathcal{R}^t \setminus \{\tilde{n}\}$
 if $f_{m\tilde{n}}^t = 1$.
 update $t \leftarrow t + 1$.
end while

3.3 Adaptation of flow size F^t

In NFBA, flow size F^t is a control parameter which influences performance. Finding the optimal set of F^t 's, however, requires enormous number of comparisons for all possible $\{F^t\}$ combinations. Hence we adapt F^t on an iteration-by-iteration basis. We consider two fixed-flow schemes including greedy allocation (i.e., NFBA-1) and bipartite matching (i.e., NFBA-max), and an adaptive flow scheme that is a generalization of these two fixed flow schemes.

3.3.1 NFBA-1 ($F^t = 1$)

In NFBA-1, we fix $F^t = 1$ for $t = 1, \dots, N$. NFBA-1 can be considered as a pure greedy algorithm whereby in each iteration, we select the subchannel that results in the largest power saving (or profit) for allocation. As a consequence, the algorithm complexity is polynomial as shown in Theorem 1 below.

Theorem 1: The complexity of NFBA-1 is $O(MN^2)$.

Proof: Finding the optimum in iteration t amounts to simply selecting the maximum-profit arc among a total of MN_t arcs. There are exactly N iterations. Hence the amount of computations is $\sum_{t=1}^N M(N+1-t)$, giving an overall complexity of $O(MN^2)$. $\square$

The motivation for NFBA-1 is from Theorem 2, which states that NFBA-1 is optimal if the channels are flat fading, i.e., every user $m \in \mathcal{M}$ has the same channel gain $g_{mn} = g_m, n \in \mathcal{N}$. Note that the channel gains still differ for different users.

Theorem 2: If the channels are flat fading, then NFBA-1 solves problem P_{orig} optimally with time complexity $O(MN)$.

Proof: For flat-fading channels, the problem degenerates to determining the number of subchannels for each user. Consider user $m \in \mathcal{M}$. It is optimal to split the rate of each user evenly among the allocated subchannels; otherwise by the convexity of the rate function, the sum energy can be reduced. With x subchannels, the sum power for user m is $p_m(x) = x(2^{\frac{R_m}{x}} - 1)/g_m$. Treating x as a continuous variable, it can be shown that function $p_m(x)$ is strictly decreasing and strictly convex for $x > 0$. For any two positive integers $v_1 < v_2$, it follows that $\delta_m(v_1) > \delta_m(v_2)$, where $\delta_m(v) \triangleq p_m(v) - p_m(v+1)$ is the power saving (profit). In other words, as the number of allocated subchannels increases, the power saving by allocating one additional subchannel strictly decreases. Denote by N_m^* and N_m the number of subchannels of user m at optimum and the number of subchannels allocated to user m in the solution returned by NFBA-1, respectively, $\forall m \in \mathcal{M}$. Suppose solution $(N_1, \dots, N_M) \neq (N_1^*, \dots, N_M^*)$, and let $\Delta = \sum_{m \in \mathcal{M}} |N_m - N_m^*|$. Note that $\sum_{m \in \mathcal{M}} |N_m| = N$. Thus if $\Delta > 0$, there must exist at least two users, say user 1 and 2, such that $N_1 > N_1^*$ and $N_2 < N_2^*$.

Consider the iteration in which user 1 is allocated its last, N_1 th channel. Denote by N_2' the number of subchannels allocated to user 2 at this stage. Because NFBA-1 applies greedy selection by power saving, we obtain $\delta_1(N_1 - 1) \geq \delta_2(N_2')$, as otherwise the channel would have been allocated to user 2 instead. Because $N_2' \leq N_2$, $\delta_2(N_2') \geq \delta_2(N_2)$. Hence, $\delta_1(N_1 - 1) \geq \delta_2(N_2)$. Because $N_1 - 1 \geq N_1^*$ and $N_2 \leq N_2^* - 1$, we obtain $\delta_1(N_1 - 1) \leq \delta_1(N_1^*)$ and $\delta_2(N_2) \geq \delta_2(N_2^* - 1)$, respectively. Using the three inequalities for profits, consequently, we arrive at $\delta_1(N_1^*) \geq \delta_2(N_2^* - 1)$. Strict inequality would contradict the optimality of $(N_1^*, \dots, N_M^*)$. Assume therefore, equality, meaning that there is an alternative optimum, in which the numbers of subchannels of users 1 and 2 are increased and decreased by one, respectively. Note that Δ is strictly reduced for the alternative optimal solutions. Considering the alternative optimum and applying the same line of argument, eventually $\Delta = 0$, hence solution $(N_1, \dots, N_M)$ is optimal. Finally, finding the optimum in each iteration amounts to selecting a maximum-profit user among M users, and there are exactly N iterations. Hence the complexity is $O(MN)$. $\square$

However, the channels are typically frequency selective. Hence in general the greedy allocation of a single subchannel in each iteration is suboptimal. Nevertheless, from Theorem 2 we expect intuitively that the performance of NFBA-1 is close to optimal if the channels are not experiencing severe frequency selectivity.

3.3.2 NFBA-max ($F^t = \min(N^t, M)$)

In NFBA-max, we set F^t to its maximum possible value in all iterations, i.e., $F^t = \min(N^t, M), \forall t$. Note that each iteration with $F^t = M$ is equivalent to solving optimal matching in a bipartite graph [18]. Accordingly, the complexity of NFBA-max follows Theorem 3.

Theorem 3: The complexity of NFBA-max is $O(N^3)$.

Proof: The flow problem reduces to maximum-weighted matching in a bipartite graph. Without loss of generality, assume N is divisible by M , and let $T = N/M$. There are exactly T itera-

tions. The numbers of nodes and corresponding arcs in iteration $t = \{1, 2, \dots, T\}$ are $(T+2-t)M$ and $(T+1-t)M^2$, respectively. By the complexity of maximum weighted matching [18], the t th iteration has $O((T+1-t)(T+2-t)M^3)$ running time, leading to an overall complexity of $O(T^3M^3) = O(N^3)$. For completeness, one has to account for the time of computing the profit values. Note that the computation in (4) does not impose the computational bottleneck. $\square$

The NFBA-max scheme resolves the potential problem of NFBA-1 because it considers the impact of channel allocation with F^t resources in the same iteration. The following theorem states a condition for optimality of NFBA-max.

Theorem 4: If $M = N$, then NFBA-max solves problem P_{orig} optimally with time complexity $O(M^3)$.

Proof: NFBA finds the optimal allocation for F^t at t . When $M = N$, $F^1 = N$ and only one iteration is required, hence NFBA-max is optimal. In this iteration, NFBA-max solves one bipartite matching problem of size $O(M)$ (after addition of dummy subchannel nodes with zero weight), from which the complexity follows. $\square$

However, NFBA-max is suboptimal when $M < N$. This is because all users are allocated $\lfloor N/M \rfloor$ or $\lfloor N/M \rfloor + 1$ subchannels in the final solution, whereas at optimum, the numbers can be highly unbalanced, especially when the target rates of users are asymmetric.

3.3.3 NFBA-adapt ($F^t = 1$ or $\min(N^t, M)$)

In NFBA-adapt, we combine the advantages of NFBA-1 and NFBA-max, by choosing F^t adaptively:

$$F^t = \begin{cases} \min(N^t, M), & \text{if } s^t \geq \tau \\ 1, & \text{otherwise} \end{cases} \quad (7)$$

where τ is a fixed threshold and s^t is a metric that reflects the uniformity of the power saving $\delta_{m\tilde{n}}^t$. We propose to use a uniformity metric defined by $s^t = \text{GM}(S^t)/\text{AM}(S^t)$. Here, $\text{GM}(S)$ and $\text{AM}(S)$ are the geometric mean and the arithmetic mean of the elements in set S , respectively, and $S^t \triangleq \{\delta_{m\tilde{n}}^t > 0, m \in \mathcal{M}, \tilde{n} \in \mathcal{R}^t\}$ denotes the set of strictly positive power savings. We note that $0 \leq s^t \leq 1$; $s^t = 1$ if and only if the power savings (elements in S^t) are the same and so $\text{GM}(S^t) = \text{AM}(S^t)$, while s^t decreases to zero as the power savings become more non-uniform. Hence, s^t describes the degree of uniformity of the power saving $\delta_{m\tilde{n}}^t$.

The strategy in (7) is devised from the observation that NFBA-max works well if the power savings are close to being uniform, while NFBA-1 is a greedy allocation that works well otherwise; therefore, we can expect that NFBA-adapt overcomes the pitfalls of both NFBA-1 and NFBA-max with an appropriate choice of threshold τ . Noting that NFBA-1 and NFBA-max are identical to NFBA-adapt with $\tau = 1$ and $\tau = 0$, respectively, we can see that the NFBA-adapt strategy is a generalization of both NFBA-1 and NFBA-max. It is difficult to find the optimum of τ analytically because the optimal τ depends on various system parameters such as R_m , g_{mn} , M , and N . Nevertheless, an exhaustive search can be conducted to determine the optimal τ numerically, as performed in Section 5, for a given simulation environment.

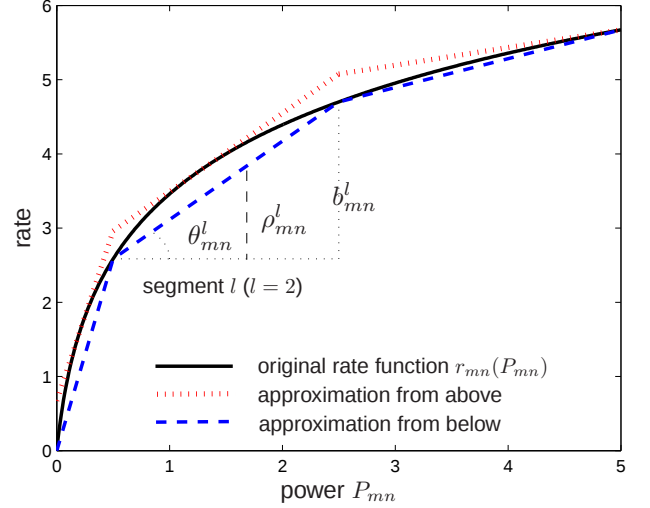


Fig. 3: For subchannel n of user m , approximation of the rate function $r_{mn}(P_{mn})$ with three segments. The second segment ($l = 2$) and its slope θ_{mn}^l and its height b_{mn}^l in the rate axis are illustrated. Rate ρ_{mn}^l varies within the segment l .

Regardless of what τ is chosen in each iteration t , the overall complexity of NFBA-adapt is bounded by $O(N^3)$ from the following theorem.

Theorem 5: The complexity of NFBA-adapt is $O(N^3)$.

Proof: When $N \geq M$, the complexity of NFBA-1 and NFBA-max are $O(MN^2)$ and $O(N^3)$, respectively, from Theorems 1 and 3. They incur N and $\lceil N/M \rceil$ iterations, respectively, so their complexity for one iteration are $O(MN)$ and $O(N^2M)$. Suppose NFBA-adapt employs a iterations of NFBA-1 and b iterations of NFBA-max. Since $a \leq N$ and $b \leq \lceil N/M \rceil$ is sufficient to allocate all subcarriers, the complexity of NFBA-adapt is $O(aMN + bN^2M) \leq O(MN^2 + N^3) \leq O(N^3)$, i.e., at most $O(N^3)$. $\square$

4 Bounds for global minimum power

In this section, we obtain upper and lower bounds for the minimum power in P_{orig} via a compact integer program, which can be solved with moderate complexity. To this end, we bound the rate function $r_{mn}(P_{mn})$ from above or below, so that we can approximate the original rate function as a concave piecewise linear function. Figure 3 illustrates the approximation of the rate function with three segments.

For the representation of approximation, we use line segment l , segment slope θ_{mn}^l , and segment height b_{mn}^l . In general, the number of segments can vary by user m and subchannel n , yet we assume that it is uniform merely for presentational simplification, i.e., $l \in \{1, \dots, L\} = \mathcal{L}$ where L is the number of segments. The amount of rate of a segment is bounded by b_{mn}^l corresponding to segment l , and the segment slope is obtained from the approximated rate function as illustrated in Fig.3. Using the approximated rate function with given $\{\theta_{mn}^l, \forall m, n, l\}$,

the compact integer programming is formulated as

$$P_{ip}^t : \arg \underset{\{f_{mn}, \rho_{mn}^l, x_{mn}, \forall m, n, l\}}{\text{minimize}} \sum_{m \in \mathcal{M}} \sum_{n \in \mathcal{N}} \sum_{l \in \mathcal{L}} \frac{\rho_{mn}^l}{\theta_{mn}^l} \quad (8a)$$

$$\text{s.t.} \quad \sum_{m \in \mathcal{M}} f_{mn} \leq 1, \quad \forall n \in \mathcal{N} \quad (8b)$$

$$\sum_{n \in \mathcal{N}} x_{mn} = 1, \quad \forall m \in \mathcal{M} \quad (8c)$$

$$\sum_{l \in \mathcal{L}} \rho_{mn}^l = x_{mn} R_m, \quad \forall m \in \mathcal{M}, \quad \forall n \in \mathcal{N} \quad (8d)$$

$$\rho_{mn}^l \leq b_{mn}^l, \quad \forall m \in \mathcal{M}, \quad \forall n \in \mathcal{N}, \quad \forall l \in \mathcal{L} \quad (8e)$$

$$x_{mn} \leq f_{mn}, \quad \forall m \in \mathcal{M}, \quad \forall n \in \mathcal{N} \quad (8f)$$

$$f_{mn} \in \{0, 1\}, \quad \rho_{mn}^l \geq 0, \quad x_{mn} \geq 0 \quad (8g)$$

Description of the variables, objective function, and constraints:

Variables:

- f_{mn} is the binary variable representing assignment that is one if subchannel n is assigned to user m and zero otherwise.
- ρ_{mn}^l is the continuous rate allocated to segment l on subchannel n for user m , which cannot exceed b_{mn}^l .
- x_{mn} is the fraction of rate of user m allocated to subchannel n .

Objective function:

- (8a) represents the total power consumption of all users on all subchannels and line segments.

Constraints:

- (8b) is for orthogonal subchannel allocation.
- (8c) ensures that the fractions of rate on all channels of each user sum over to 100%.
- (8d) states that for each user the rate to be supported on a channel (the right-hand side) equals to the sum of the rates allocated to the segments (the left-hand side).
- (8e) simply sets the upper limit of rate that can be supported by each segment. Since the inverse function of rate function is convex, the first segment will be filled first at optimum, before moving to the next. In other words, the optimal solution will allocate power to segment l_2 only after saturation of power allocation on all segments l_1 with $l_1 < l_2$.
- (8f) connects two sets of variables. In effect, it ensures that a user m can use subchannel n for serving (any fraction of) its rate demand, only if the subchannel n is allocated to user m .

Assignment mapping:

- $\mathcal{N}_m \leftarrow \bigcup_{n \in \mathcal{N}} \{n | f_{mn} = 1\}, \quad \forall m.$

Note that variable x_{mn} can be eliminated by variable substitution using (8d). Formulation (8) is compact, namely, the numbers of variables and constraints are both polynomial in problem size, so that an off-the-shelf integer solver is applicable.

5 Numerical results

In this section, we evaluate total power consumption of the proposed network-flow-based algorithms. Power gap (%) from the lower bound of the minimum power is also shown in Table 1. More results regarding to NFBA-adapt are provided in Fig. 4.

5.1 Scenarios

We assume Rayleigh fading channels with unit variances. We consider a small-scale system with $(M, N) = (3, 8)$ and a large-scale system with $(M, N) = (20, 50)$. For each system, we use either symmetric or asymmetric target rates. In symmetric target rates, all users target rates are identical as $R_m = 1, \forall m$. In asymmetric target rates of the case when $(M, N) = (3, 8)$, $R_2 = 2R_1$ and $R_3 = 4R_1$, and the case when $(M, N) = (20, 50)$, $R_{m_1} = R_1$ for $m_1 = 2, \dots, 8$, $R_{m_2} = 2R_1$ for $m_2 = 9, \dots, 18$, and $R_{19} = R_{20} = 4R_1$.

5.2 Schemes

Numerical results are obtained for NFBA-1, NFBA-max, and NFBA-adapt as described in Section 3. For small-scale systems, we also consider the NFBA-opt scheme, where we optimize NFBA with an exhaustive search of all possible combinations of $\{F^t\}$. In NFBA-adapt, we set the adaptation threshold τ to 0.7, which yields the closest performance to NFBA-opt in small-scale system. Moreover, the upper and lower bounds of the minimum power are obtained by solving the integer program in Section 4 (we shall use the lower bound for subsequent comparisons); for small-scale systems, we also obtained the exact minimum power via an exhaustive search. For the sake of comparison, we include the amplitude-craving-greedy (ACG) and rate-craving-greedy (RCG) algorithms in [9]. To do this, we modify the ACG and RCG algorithms with a continuous rate function and an exact water-filling solution.

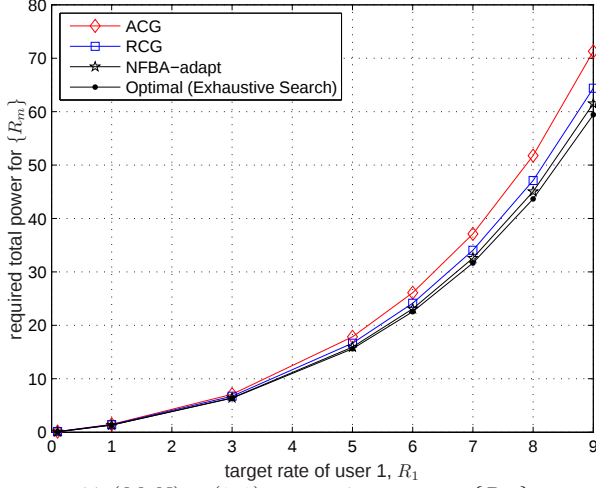
5.3 Observations

The numerical results are shown in Table 1 when $R_1 = 1$. From the last column of Table 1, the lower and upper bounds are tight to less than 0.33%. (In defining the approximation for rate function in (8), we have ensured that the error is at most 1% for the entire rate range of every user and channel). This allows for comparisons of the various schemes to the global optimum to within 0.33% error. In the table, the additional total power required compared to the lower bound is given as a percentage in boldface. For $(M, N) = (3, 8)$, NFBA-adapt achieves almost identical performance to that of NFBA-opt, which verifies that the adaptation in (7) works well. With symmetric and asymmetric target rates, the optimality gaps of NFBA-adapt are no more than 0.64% and 3.68%, respectively. Actually, the performance gap between NFBA-adapt and real global optimum from the exhaustive search is only 0.1%. Similar observations apply to $(M, N) = (20, 50)$. In fact, with asymmetric target rates, the gap is smaller compared with small-scale systems: the performance gap to global optimum is only 1.98%. Hence, we can conclude that the proposed NFBA-adapt can achieve a near-optimal solution even for large-scale systems.

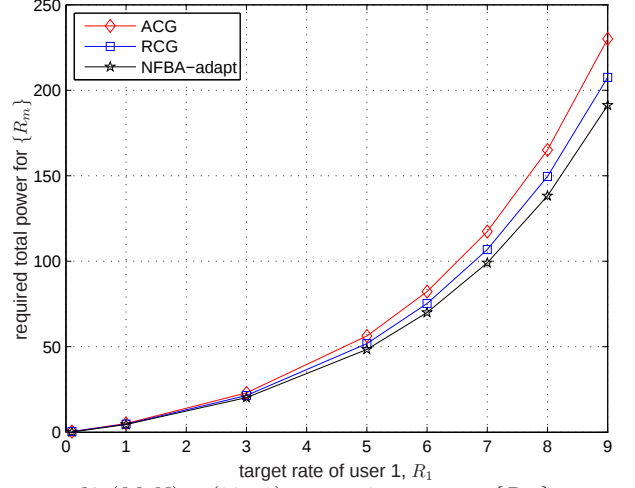
In Fig. 4, we compare total power consumption over various R_1 . Figures 4(a) and (b) show the results for the small-scale and

Table 1: Additionally Required Total Power (% in Boldface) Compared to The Lower Bound (Fix $R_1 = 1$).

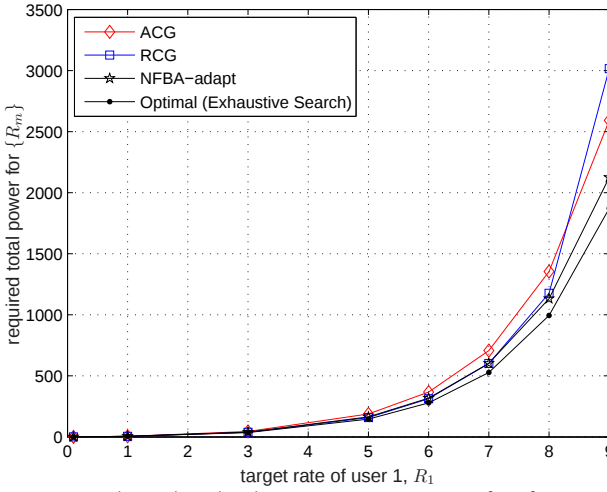
		ACG	RCG	NFBA-1	NFBA-max	NFBA-adapt	NFBA-opt	Exhaustive search	min. power lower bound	min. power upper bound
Sym. target rate	$(M, N) = (3, 8)$	1.45865	1.40492	1.38192	1.33334	1.33261	1.33187	1.33110	1.32807	1.33465
		9.83%	5.79%	4.06%	0.40%	0.34%	0.29%	—	$\pm 0.24\%$ from avg.	
	$(M, N) = (20, 50)$	5.04133	4.77191	4.58144	4.53945	4.53803	—	—	4.50932	4.53790
Asym. target rate	$(M, N) = (3, 8)$	5.57202	4.95168	5.14846	4.96429	4.84466	4.82482	4.68700	4.67292	4.70441
		19.24%	5.97%	10.18%	6.24%	3.68%	3.25%	—	$\pm 0.31\%$ from avg.	
	$(M, N) = (20, 50)$	11.48617	10.65136	10.336865	10.35249	10.11387	—	—	9.91701	9.98222
		15.82%	7.40%	4.55%	4.39%	1.98%	—	—	$\pm 0.33\%$ from avg.	



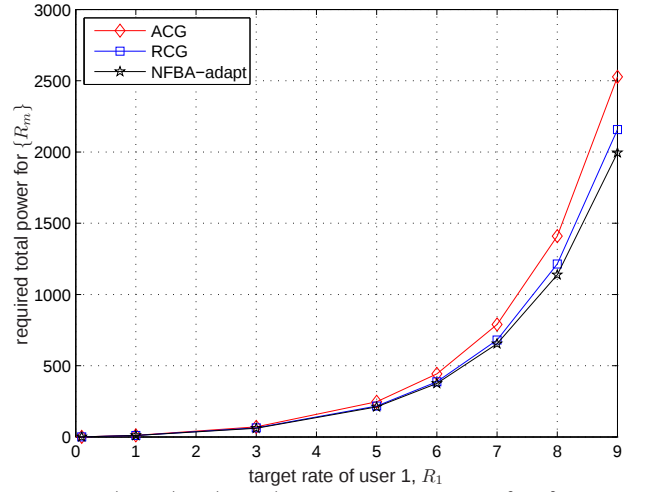
(a) $(M, N) = (3, 8)$, symmetric target rates, $\{R_m\}$



(b) $(M, N) = (20, 50)$, symmetric target rates, $\{R_m\}$



(c) $(M, N) = (3, 8)$, asymmetric target rates, $\{R_m\}$



(d) $(M, N) = (20, 50)$, asymmetric target rates, $\{R_m\}$

Fig. 4: Power consumption comparison for different target rates. In the x-axis, we vary the target rate of user 1, R_1 , and vary the rates of other users as follows: (a) $(M, N) = (3, 8)$ with symmetric target rates where $R_1 = R_2 = R_3$; (b) $(M, N) = (20, 50)$ with symmetric target rates where $R_m = R_1$ for $m = 2, \dots, 20$; (c) $(M, N) = (3, 8)$ with asymmetric target rates where $R_2 = 2R_1$ and $R_3 = 4R_1$; (d) $(M, N) = (20, 50)$ with asymmetric target rates where $R_{m_1} = R_1$ for $m_1 = 2, \dots, 8$, $R_{m_2} = 2R_1$ for $m_2 = 9, \dots, 18$, and $R_{19} = R_{20} = 4R_1$.

large-scale systems, respectively, with symmetric target rates, while Figs 4(c) and (d) show the results for the small-scale and large-scale systems, respectively, with asymmetric target rates. For the small-scale systems, the exhaustive search (optimal) is also included. From the results, we can verify that the NFBA-adapt algorithm achieves the nearest performance to the global optimal for all simulation setup.

6 Conclusions

We introduced a network-flow-based representation of resource allocation problem and proposed a heuristic network-flow-based algorithm (NFBA) requiring polynomial complexity with a flow size adaptation strategy. Furthermore, a compact integer programming problem was formulated to provide tight lower

and upper bounds of the minimum power. Numerical results verified that the proposed NFBA with flow size adaptation can achieve near optimal performance and that the bounds are very tight.

7 References

1. Sadr, S, Anpalagan, A., and Raahemifar, K.: 'Radio resource allocation algorithms for the downlink of multiuser OFDM communication systems', IEEE Commun. Surveys & Tutorials, Third Quarter 2009, **11**, (3), pp. 92–106.
2. Bohge M, Gross J., Wolisz A., and Meyer M.: 'Dynamic resource allocation in OFDM systems: An overview of cross-layer optimization principles and techniques', IEEE Network, Jan./Feb. 2007, pp. 53–59.
3. Yen, Y.-S., Chan, Y.-K., Chao, H.-C., and Park, J. H.: 'A genetic algorithm for energy-efficient based multicast routing on MANETs', Computer Communications, Mar. 2008, **31**, (4), pp. 858–869.
4. Huang, P.-H., Gai, Y., Krishnamachari, and B., Sridharan, A.: 'Subcarrier allocation in multiuser OFDM systems: Complexity and approximability', Proc. IEEE Wireless Commun. Network Conf. (WCNC), Sydney, Australia, Apr. 2010.
5. Yuan, D., Joung, J., Ho, C. K., and Sun, S.: 'Tractability and computational aspects of optimal resource allocation in OFDMA systems', submitted to IEEE Trans. Vehicular Technology, <http://arxiv.org/abs/1112.2372>, accessed January 2012.
6. Wong, C. Y., Cheng, R. S., Letaief, K. B., and Murch, R. D.: 'Multiuser OFDM with adaptive subcarrier, bit, and power allocation', IEEE Journal on Selected Areas in Communications, Oct. 1999, **17**, (10), pp. 1747–1758.
7. Rhee, W. and Cioffi, J. M.: 'Increase in capacity of multiuser OFDM system using dynamic subchannel allocation', Proc. IEEE Vehicular Technology Conference (VTC), Tokyo, Japan, May 2000, pp. 1085–1089.
8. Seong, K., Mohseni, M., and Cioffi, J. M.: 'Optimal resource allocation for OFDMA downlink systems', Proc. IEEE International Symposium on Information Theory (ISIT), Seattle, USA, Sep. 2006. pp. 1394–1398.
9. Kivanc, D., Li, G., and Liu, H.: 'Computationally efficient bandwidth allocation and power control for OFDMA', IEEE Trans. Wireless Commun., Nov. 2003, **2**, (6), pp. 1150–1158.
10. Feiten, A., Mathar, R., and Reyer, M.: 'Rate and power allocation for multiuser OFDM: An effective heuristic verified by branch-and-bound', IEEE Trans. Wireless Communications, Jan. 2008, **7**, (1), pp. 60–64.
11. Liu C., Schmeink A., and Mathar R.: 'Power allocation for broadcasting in multiuser OFDM systems with sublinear complexity', Proc. IEEE International Conf. Commun. (ICC), Dresden, Germany, Jun. 2009.
12. Liu T., Yang C., and Yang L.-L.: 'A low-complexity subcarrier-power allocation scheme for frequency-division multiple-access systems', IEEE Trans. Wireless Commun., May 2010, **9**, (5), pp. 1571–1576.
13. Hu Y., Huang Y., Yang L., and Zhou J.: 'Energy-efficient resource allocations in multi-user OFDMA systems', Proc. IEEE Wireless Communications and Signal Processing (WCSP), Nanjing, China, Nov. 2011.
14. Ho C. Y. and Huang C.-Y.: 'Energy-efficient 2-D resource allocation with fairness constraints for OFDMA networks', Proc. IEEE Vehicular Technology Conference (VTC), San Francisco, USA, Sep. 2011.
15. Zaki A. N. and Fapojuwo A. O.: 'Optimal and efficient graph-based resource allocation algorithms for multiservice frame-based OFDMA networks', IEEE Trans. Mobile Computing, Aug. 2011, **10**, (8), pp. 1175–1186.
16. Jang J. and Lee K. B.: 'Transmit power adaptation for multiuser OFDM systems', IEEE Journal on Selected Areas in Communications, Feb. 2003, **21**, (2), pp. 171–178.
17. Bingham, J. A. C.: 'Multicarrier modulation for data transmission: an idea whose time has come', IEEE Commun. Mag., May 1990, **28**, (5), pp. 5–14.
18. Ahuja, R. K., Magnanti, T. L., and Orlin, J. B.: 'Network flows: theory, algorithms, and applications', Prentice-Hall, 1993.