Optimisation of Functionally Graded Lattice Structures using Isostatic Lines

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Abstract

Functionally graded lattice core structures show a gradual and localised variation in their mechanical properties with the aim of improving structural performance whilst minimising weight. We present a novel approach to generate optimised functionally graded lattice core structures. Firstly, topology optimisation is performed to return the optimal core density distribution to minimise the structure's compliance subject to a mass constraint. A series of isostatic lines are then constructed with respect to the local principal stresses to generate a lattice structure spatially graded with respect to lattice cell size, aspect ratio and orientation. To validate this novel approach, optimisation is performed on a sandwich core structure subject to three point bending. Experimental tests confirm the greatly improved stiffness and strength properties (101% and 172% respectively) of the core as a result of spatially grading the lattice cells when compared to a benchmark core with uniform cell size of the same density. The new approach also significantly outperforms lattice structures with graded diameters as optimised by state-of-the-art commercial software packages. Non-dimensional core performance indices are formulated to express the relative specific stiffness and strength properties of the core for the three design approaches.

Keywords

Topology optimisation, lattice structures; finite element analysis; functionally graded materials; sandwich structures.

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1. Introduction

Cellular core structures play a vital role in the design of lightweight structural components for a wide range of applications [1]. Conventional cellular core manufacturing techniques such as expansion, corrugation and moulding place significant restrictions on the available core geometries that can be realised, and in turn this restricts the available design space [2, 3]. However, recent advances in the additive manufacture of both polymeric and metallic components have opened up a wide design space largely free of manufacturing constraints or limitations related to design complexity. These increased design freedoms offer the potential of achieving highly mass efficient designs by manufacturing cores with tailored cell geometries that are close to the idealised optimal solutions [4].

Functionally graded structures have recently emerged as a means to locally tailor mechanical properties with the aim of improving structural performance [5]. Continuous variation in mechanical properties to generate structurally efficient designs can also be observed in nature, for example, in the hierarchical structures inside bamboo or bone. Bamboo demonstrates a densification in material towards its outer surface in order to efficiently withstand bending moments [6]. Similar property changes can be seen in the cross-sections of human femurs [7]. Continuous variation in properties is important in a range of structural applications to minimise stress concentrations. The ability to locally and continuously tailor the stiffness in a material has also attracted research interest in the design of fibre reinforced composites where controlling local fibre orientations using curved fibre trajectories has been shown to be an effective method of realising optimised lightweight structures capable of withstanding high buckling loads [8]. Similarly, hybrid joints with two adjoining adherends with different stiffness and thermal expansion coefficients require a gradual transfer in load without high localised stress gradients [9].

A functionally graded sandwich core structure is also useful in applications for core materials subject to impact where the stiffness profile during crushing can be tailored [10]. This was demonstrated experimentally with 3D printed truss-like lattice structures whereby cells can be designed to progressively collapse under compressive loads prior to final densification [11]. 3D printed truss-like lattice structures have also been demonstrated to have excellent stiffness and strength performance when used as a core material in sandwich structures [12, 13, 14]. However, there is the potential for further improvements of these mechanical properties as a result of
functionally grading lattice structures [15] where there is a variation in in-plane stress through the thickness of the structure. These types of core structures are the focus of our present work.

Lattice structures consist of a large number of trusses arranged in spatial patterns with selected unit cell design. The cells’ optimum arrangement has been the focus of numerous research studies. Very early work in the field of structural optimisation became known as Michell structures [16, 17]. Michell developed a criterion for a minimum mass truss structure subject to loading. The criterion states that all structural members must be strained by the same amount. The required deformation requires that members are arranged in a frame that is aligned with principal stress directions. The requirement to follow principal directions results in curves of orthogonal systems. A limitation of this early analytical work is that its implementation is restricted to simple geometries. In the years since, various computational methods have been developed, including size and topology optimisation approaches, which are capable of optimising a much broader range of problems [18]. Size optimisation can be used to parametrically control, for example, the individual diameters of trusses in any lattice configuration in response to external load cases. However, the question of the initial optimum truss arrangement is not addressed with this approach. In contrast, topology optimisation methods are generally concerned with achieving binary solid-void solutions. With topology optimisation the less stressed elements in a finite element model are assigned an artificially softer material through penalty methods. In this way, holes in the structure can be approximated as regions of low density below a defined threshold, and an optimal material layout can be determined. Topology optimisation can therefore allow greater design freedom than size optimisation. Various topology optimisation approaches have been developed over recent years [19, 20, 21, 22] including density-based, hard kill, level-set [23] and bio-inspired methods such as genetic algorithms. Instead of binary solutions, topology optimization may also be used to determine optimum and gradually-changing stiffness responding to applied loads.

In practical designs, a continuously changing stiffness covering a large stiffness range and intermediate densities is difficult to achieve. Recently, advances in metal and polymer 3D printing technology and related topology optimisation approaches have resulted in new approaches to achieve functional graded structures. Through the development of a level-set method for topology and material property optimisation, Xia and Wang [24] demonstrated that an optimum design for functionally graded structures can be achieved assuming a continuous medium that
can vary its stiffness through the dispersion of two materials. An alternative approach is to design lattice structures with smoothly varying porosity to realise optimal stiffness gradients [25, 26].

In this paper, we demonstrate a novel concept for the generation and optimal design of functionally graded lattice core structures for stiffness and strength [27]. Using a combined methodology of topology optimisation and size optimisation, we identify principal stress directions in a topology optimised structure and align truss geometries with optimised diameters with the established isostatic stress lines. For validation purposes, the core geometry for a sandwich structure subject to three point bending is additively manufactured using body centred cubic (BCC) lattice cells where the size, shape and orientation of each lattice cell, as well as the lattice member’s cross-sectional diameter, are optimised. We clearly demonstrate that our novel approach significantly increases bending stiffness and strength in comparison to the uniform and diameter graded lattice structures (based on state-of-the-art commercial software lattice optimisation tools) of a similar bulk density.

2. Optimisation Methodology

Topology optimisation is a powerful tool in the development of innovative designs that are lightweight and structurally efficient. Since topology optimisation tends to result in complicated geometries that are impossible to realise using traditional manufacturing techniques, it is a particularly well suited design tool for the development of 3D printed structures with much enhanced manufacturing freedom. In comparison, the limitations of traditional manufacturing techniques such as casting or machining tend to result in lost design efficiency as they cannot realise the target optimal geometry.

In this current work we make use of Altair OptiStruct 13.0 for topology and size optimisation. The software has unique analysis capabilities related to the design and optimisation of 3D printed lattice structures [29]. The formulation for the density-based topology optimisation problem considered in this paper is given in Equation (1). The formulation is based on linear static finite element analysis.

\[
\begin{align*}
\text{minimise: } & \quad f(\rho, U) = U^T K U \\
\text{subject to: } & \quad K(\rho)U = F(\rho)
\end{align*}
\] (1)
\[ g(\rho, \mathbf{U}) = V - V_f \leq 0 \]
\[ 0 \leq \rho \leq 1 \]

In Equation (1) \( f \) is the objective function, \( \rho \) is a vector containing the density design variables, the displacement vector is given by \( \mathbf{U} \) and the global stiffness matrix is \( \mathbf{K} \). In this work the objective is to minimise structural compliance, which is defined as \( \mathbf{U}^T \mathbf{K} \mathbf{U} \). To satisfy equilibrium, the stiffness matrix and displacement matrix are related to the force vector \( \mathbf{F} \). The optimisation problem is constrained by the function \( g \), which links the material volume fraction in the design space \( V \) and the allowable volume fraction upper bound, specified in our case as \( V_f = 0.25 \).

Porous materials, such as lattice structures, typically exhibit lower stiffness per unit volume compared to the equivalent fully dense material. A semi-empirical relationship between lattice stiffness and solid stiffness is implemented in OptiStruct using

\[ E = (\rho/\rho_0)^n E_0 \]  

(2)

where \( E_0 \) is the Young’s modulus and density of the solid material respectively. The exponent \( n \) depends on a particular cell geometry but is typically in the region of \( n = 2 \) for bending-dominated behaviour and \( n = 1 \) for stretch dominated behaviour [28]. In this work we apply \( n = 1 \) since the optimal solution will favour a higher proportion of lattice zones, and less voids or solid regions, being formed.

There are two functionally graded lattice design methodologies considered in this current study: diameter grading and spatial grading. Diameter grading is performed as part of the state-of-the-art OptiStruct capabilities in Version 13.0 without further modifications. Spatial grading is achieved through the development of in-house user subroutines linked with OptiStruct. Further details are given in the following sections.

2.1. Diameter graded lattice structures

OptiStruct optimises lattices using a two-step approach (see Figure 2, right side only). Firstly, topology optimisation is applied to a 3D solid element model resulting in an optimum density graded material. The porosity of this hypothetical material is able to vary between 100% (fully densified) and 0% (empty space). The porous zones are then explicitly transformed into BCC
lattice structures consisting of beam elements of varying cross-sectional diameters. It should be noted that the optimisation methodology employed here is in principle independent of the unit cell type. In a second step, the lattice structure is subject to size optimisation to achieve the target cell density. This enables the diameter of each lattice member to be individually optimised. 3D additive manufacturing constraints can also be applied at this second step in the form of placing an upper and lower bound on feasible lattice diameters.

The edge length dimension of each lattice cell \( l \) is determined by the initial mesh used for topology optimisation and is therefore not a variable in this optimisation procedure. The lattice cells considered in this study are BCC as this cell topology has relatively linear and isotropic stiffness properties. The manufacturing constraints imposed during size optimisation limit the lattice beam diameters to a range of \( 1/10 \leq \Phi/l \leq 2/5 \), see Figure 1. This range of diameters results in a range of densities that closely approximate the optimal densities shown in Figure 1. A lower bound on the diameter is required as each 3D printer has a minimum printing resolution. An upper bound is required as large values of \( \Phi/l \) result in the formation of fused cells during 3D printing.

**2.2. Spatially graded lattice structures**

The ability to optimise lattice cell size, aspect ratio and orientation is not available in OptiStruct 13.0 since the cell size is mesh-dependent and not an optimisation variable. To introduce this new functionality, an additional user defined routine has been developed for this research study using MATLAB. This approach introduces two additional steps prior to diameter grading of the lattice and final size optimisation, as shown in Figure 2. To spatially grade the lattice cells, topology optimisation is used at first with a regular mesh to determine the optimal density distribution and bulk stress state within the core. This step is the same for the diameter grading procedure. Data is then output by OptiStruct for the in-plane stress components (\( \sigma_x \), \( \sigma_y \) and \( \sigma_{xy} \)) within the topology optimised core. This data enables the maximum principal stresses to be determined in addition to their orientation (\( \sigma_1 \), \( \sigma_2 \) and \( \theta \), respectively) using the equations of statics or the Mohr’s circle approach [30]. An optimal spatially graded mesh is then produced in MATLAB based on this stress data using the newly developed isostatic line method, which is described in more details in the next section. The optimal spatially graded mesh is finally re-analysed in OptiStruct using the same two-step procedure used for the diameter graded lattice.
2.3. Construction of Isostatic Lines for Spatially Graded Lattice Cells

With a given set of stress data in a 2D plane, which is obtained numerically in this current work via finite element analysis, it is possible to construct two orthogonal sets of isostatic lines that describe the maximum and minimum principal stress trajectories within the density-optimised core. Since these isostatic lines are aligned with the principal stress trajectories, they are by definition free of shear stress. A general analytical method for the construction of isostatic lines is not known so a numerical approach is adopted in this work. The numerical construction of isostatic lines in this work is a development on [31, 32] where stress data is numerically integrated along the stress trajectories using Euler’s method.

The method is implemented using MATLAB and interfaced with OptiStruct input and output files in order to generate spatially graded lattice meshes. The first step is to select a starting point to commence integration; a corner on the core’s boundary usually proves a convenient point to place an initial isostatic line. Once a starting point is defined, the MATLAB script will draw orthogonal lines based on the local principal stresses \((\sigma_1, \sigma_2)\) until they reach a point outside the core’s domain. The angle \(\theta\) defining the principal stress direction at a point along with the derivation of the principal stresses from the global stress components are defined in [30]. The orientation of the maximum principal stress is found using:

\[
\theta = \frac{1}{2} \tan^{-1} \left( \frac{\tau_{xy}}{\sigma_x - \sigma_y} \right)
\]  

(3)

The stress components at a given point in terms of the global coordinate system \(\sigma_x, \sigma_{xy}\) and \(\tau_{xy}\) are found by interpolating the stress data from the finite element analysis. An isostatic line can then be traced by incrementally moving by distance \(ds_1\) in the direction of \(\theta\) or by moving a distance \(ds_2\) orthogonal to the direction of \(\theta\) and calculating the relative movements in the global coordinate system

\[
\begin{align*}
\sigma_1: & \quad dx = ds_1 \cos \theta, \quad dy = ds_1 \sin \theta \\
\sigma_2: & \quad dx = -ds_2 \sin \theta, \quad dy = ds_2 \cos \theta
\end{align*}
\]

(4)
Once the first two isostatic lines are defined \((\varphi_i, \psi_j)\) it is then possible to calculate the trajectories of successive isostatic lines \((\varphi_{i+1}, \psi_{j+1})\). The positions of the successive isostatic lines can be determined numerically using the following conditions:

\[
F = \left| \sum \sigma_1 \, d \sigma_2 \right| = \left| \sum \sigma_2 \, d \sigma_1 \right|
\]

where \(F\) is a constant that determines the relative spacing of the isostatic lines and has dimensions of force per unit thickness, see Figure 3. It has been previously noted that although isostatic lines indicate the flow of local stress, they do not strictly enclose regions in which the load is constant [33]. However, applying the approximate conditions in Equation (5) provides a convenient and simple means to space isostatic lines where no exact spacing definition is available. It should be noted that this new approach results in lattice designs resembling biological structures. Interestingly, it has for example been observed that the cellular structures within bone (trabeculae) are also oriented with respect to the principal stress directions within the femoral head [7].

3. Optimised Lattice Designs

3.1. Proof-of-Concept Demonstrator

The focus of this study is the optimal design of flat sandwich structures subject to three point bending whereby a lattice core is used in combination with an upper and lower face sheet and a centrally applied load is applied between two simply supported boundary conditions. This application has been selected for several reasons. Firstly, lattice structures are generally used for core materials which are typically of low density comparative to the facing material when designed to withstand externally applied bending moments. Secondly, existing ASTM standards exist to determine flexural and shear stiffness of sandwich structures in three point bending [34, 35, 36]. And finally, a set of optimal performance indices have already been established by Ashby [37] for the design of lightweight panels in bending in terms of both strength and stiffness. These performance indices quantify the performance of the lattice core as if it were a homogeneous bulk material.
The geometry of the three point bending specimen and its topology optimised core density distribution found using OptiStruct are shown in Figure 4. The dimensions in Figure 4 have been defined in terms of the lattice cell size $l$. Each lattice cell is a cube with volume $l^3$. The core is six lattice cells thick ($b = 6l$) and 30 lattice cells in length ($L = 30l$). To reduce computational expense only half of the beam length is modelled due to symmetry about the loading plane and the beam is only one lattice row wide ($w = l$). Modelling a single row of lattice cells also ensures that the analysis simplifies to the case of two-dimensional plane stress. The thicknesses of the face sheets, $t$, are $l/5$ each and end face sheets of thickness $l$ are also included since the topology optimisation returns zero-density results in regions where bending moments are not applied.

The objective of the topology optimisation is to minimise the beam compliance subject to a 25% total volume constraint (or 75% mass loss) placed on the core material. This volume constraint is set in a mostly arbitrary manner for this demonstrator problem, but works well with the manufacturing constraints. The optimised density distribution shown in Figure 4 reveals two main features. The first is that the highest density of close to 100% occurs near the face sheets at the mid-span. This is an intuitive result since these regions are subject to the highest tensile and compressive stresses. The second feature is an increase in density at the supports where reaction forces are applied to the structure. In other regions the density varies with a minimum value of just under 5%.

### 3.2. Optimised Design Benchmarking

BCC unit cell structures are used for all three designs. The two optimised core designs as a result of applying the diameter grading and spatially grading methods are presented in Figure 5. In addition to the two optimised designs, a ‘benchmark’ design was also additively manufactured for comparison that has a uniform density lattice structure comprising of beams with diameter equal to $\theta/l = 0.5$. Each core is designed to have a density of approximately 25% relative to the baseline material and 180 cells per row in the $z$-direction. The spatially graded core structure has a total of 186 cells per row: 160 cells have quadrilateral cross-sections and 26 have triangular cross-sections.

The distribution of the isostatic lines calculated using MATLAB are shown in Figure 6a where they are coloured in red and blue to denote which lines are subject to tensile or compressive forces respectively. This distribution of isostatic lines was generated using a non-dimensional force per
unit thickness constant equal to \( F = 0.164/P \). This constant value was selected in order to generate approximately 180 lattice cells along the length of the beam; consistent with the benchmark and diameter graded specimens. Lattice cells with triangular cross-sections can be considered as half cells as these cells are typically generated when the isostatic lines intersect the boundary of the core domain and it is not feasible to generate a full cell. This results in a potential change in failure mode in this part of the structure as will be discussed later. It can also be noted on the core boundary in Figure 6a that some nodes are very close to one another and that some very small cells are also generated at the boundary of the core domain. In such cases, these nodes are either merged or deleted to avoid the formation of excessively small cells.

The rationalised spatially graded core design after the removal and combining of these unnecessary nodes is shown in Figure 6b. Curvilinear isostatic lines are discretised into straight beam segments when generated in OptiStruct. OptiStruct also introduces central nodes in each hexahedron cell to form the diagonal members of the BCC topology. The diagonal members are included to satisfy Maxwell’s stability criterion [38]. The non-dimensional axial stress distribution after final size optimisation is also shown in Figure 6b. It can be seen that the resultant axial stress distribution has the desired distribution of tension and compression forces reflecting the original isostatic line distribution. It can also be seen that the magnitude of the stresses in the lattice cross members are comparatively low since the cells are by definition orientated to minimise shear stresses. Shear stresses are introduced mainly due to the discretisation of curvilinear lines into straight beam segments.

### 3.3. Manufacture and Experimental Validation

The optimised finite element models are exported from OptiStruct in solid geometry format (.stp) and then converted to stereolithography format (.stl) using SolidWorks prior to printing. No additional smoothing step is applied to the joint connections. The three beam demonstrators were additively manufactured from VisiJet CR-WT ‘ABS-like’ material using a 3D Systems Projet 5500X printer. The material has a flexural modulus of \( E = 1.7 \) GPa, a density of \( \rho = 1.17 \) g/cm\(^3\) and a flexural strength of \( \sigma_y = 65 \) MPa [39]. The manufactured benchmark lattice cell has an edge length of \( l = 5\) mm, resulting in a minimum lattice beam diameter of \( \bar{O} = 0.5 \) mm and a core with external dimensions \( L = 150 \) mm and \( b = w = 30 \) mm. The Projet 5500X printer has a resolutions of \( 375 \times 375 \times 790 \) DPI (67 \( \mu \)m \times 67 \( \mu \)m \times 32\( \mu \)m) in the \( x, y \) and \( z \) directions respectively.
The beam dimensions were selected to closely conform to the relevant ASTM standards related to the stiffness and strength testing of sandwich structures [34, 35, 36]. However, two geometric specifications were not possible to conform to using these types of lattice designs. Firstly, the standards specify a minimum beam width of 75 mm. It was decided that this would consume a prohibitive amount of material and instead finite width correction factors are applied in this study. Secondly, the standards specify overhangs of at least 20 mm past each of the supports in the length-wise direction. The absence of bending moments past the supports resulted in the formation of zero density cores in these regions after optimisation. Instead, 5 mm thick solid face sheets are added at the beam ends.

Three point bend tests were performed with an Instron 5982 machine using a 10 kN load cell under quasi-static conditions (see Figure 7). In addition to cross-head displacement and load cell data, the mid-point deflection of the lower surface was also recorded using an extensometer with a 25 mm gauge length. A digital image correlation (DIC) system was also used to record the displacement field of the sandwich structure’s side wall. A black and white speckle pattern was applied to the side wall to provide sufficient contrast for the DIC system to detect changes in displacement.

4. Measuring Lattice Core Performance

4.1. Calculation of Core Shear Modulus and Young’s Modulus

As the mechanical properties of the sandwich facings are known, the flexural stiffness and the transverse shear rigidity of the three core designs can be determined from the structural compliance. Since both the finite element analysis and experimental results have a finite width, it is necessary to scale the material properties calculated to represent that of a comparative infinitely wide beam. This is done by applying a weighting function to the core Young’s modulus, shear modulus and yield strength, which is proportional to the relative change in core volume between an infinitely wide core and that of a core with finite width, see Table 1. These correction factors were based on the volumes of the stereolithography models generated for 3D printing compared to the volume of an idealised repeating structure. In the case of the experimental test, these correction factors are close to unity since the experimental samples are 6 cells wide. However, the finite element models are only one cell wide and so require larger volumetric
correction factors. In addition, the finite element model approximates each lattice member as having a simple cylindrical geometry and does not accurately calculate the volume to include self-intersections at each of the lattice nodes.

Another important consideration to account for is the transverse shear deformation in the analysis of sandwich structures subject to three point bending due to their relatively large thickness-to-length ratio. The compliance of a simply supported beam subject to a centrally applied load, under combined bending and transverse shear deformation, can be expressed as

\[
\frac{\delta}{P} = \frac{L^3}{48(EI)^{eq}} + \frac{L}{4\kappa(AG)^{eq}}
\]

where \((EI)^{eq}\) and \((AG)^{eq}\) are the equivalent flexural stiffness and shear rigidity respectively and \(\delta\) and \(P\) are the mid-span deflection and total applied load, respectively. The second term in Equation (6) describes the transverse shear deformation of the sandwich structure and can be neglected only if the ratio \((EI)^{eq}/\kappa L^2(AG)^{eq} \ll 1\), see Table 2. The symbol \(\kappa\) is known as the shear coefficient, the value of which depends upon geometry. For rectangular sections, a value of \(\kappa = 5/6\) is often used [40]. The shear rigidity term in Equation (6) was determined from finite element analysis by subjecting the three beam designs to pure shear. In the case of pure shear the flexural rigidity term in Equation (6) can be omitted. Since the resultant values in Table 5 are relatively high, it was decided to include the influence of transverse shear deformation as a precaution in any subsequent analysis.

By convention, for sandwich beam analysis the transverse shear rigidity of the face sheets is omitted in the calculation of \((AG)^{eq}\) as it assumed that the face sheets are subject to pure Euler bending:

\[
G^c = \frac{(AG)^{eq}}{wb}
\]

where \(w\) is the beam width and \(b\) is the height of the core. Once the shear rigidity of the core is determined, the flexural rigidity of the structure can then be determined from combined bending/shear compliance of the finite element model using the full form of Equation (6). The
effective Young’s modulus of the core can then be found from the flexural rigidity of the structure as follows:

\[
E^c = \frac{12(EI)^{eq}}{wb^3} - \frac{2Et^3}{cb^3} \left(1 + \frac{3}{t^2}(b + t)^2\right)
\]  

(8)

**4.2. Calculation of Core Performance Indices**

A series of performance indices are formulated in this section to allow for a quantitative comparison between the various core designs presented. This approach was first proposed by Ashby [37] as a means of formulating quantifiable material selection criteria. Although each structure in this study is manufactured from the same material, the various lattice designs can be treated as a means of changing the bulk material properties of the cores. The objective in this study is to reduce the core mass described by

\[
m = Lwb\rho^c
\]  

(9)

The length \(L\), width \(w\) and load \(P\) are specified whereas we assume the thickness \(b\) to be a free design variable. The mass can be reduced by reducing \(b\) but at the expense of core stiffness and strength. Similarly to Equation (6), the maximum allowable compliance of the core \(C\) can be written in terms of combined flexural and shear deformations

\[
C \geq \frac{L^3}{4wb^3E^c} + \frac{L}{4kwbg^c}
\]  

(10)

where the core’s area and second moment of area are given by \(wb\) and \(wb^3/12\) respectively. By only considering the shear deformation contribution in Equation (10), re-arranging in terms of core thickness and substituting in Equation (9), it is possible to derive an expression which relates the mass of the core in terms of core shear modulus

\[
m \geq \left(\frac{1}{4kC}\right) L^2 \left(\frac{\rho^c}{G^c}\right).
\]  

(11)

From Equation (11) it is possible to see that the shear performance index to create a light, but stiff core is one with minimum \(\rho^c/G^c\); all other quantities are specified by the design’s geometry
and loading conditions. Similarly, by only considering the flexural term in Equation (10), rearranging in terms of core thickness and substituting into Equation (9) yields an expression for the core mass in terms of core Young’s modulus

\[ m \geq \left( \frac{w^2}{4C} \right)^{1/3} L^2 \left( \frac{\rho^c}{\sqrt[3]{E^c}} \right) \]  

(12)

In this case it is desirable to have a core material with minimal \( \rho^c / \sqrt[3]{E^c} \) in order to create a light, but stiff core. For the case of core yield strength \( \sigma_y^c \), the core must not fail under load \( P^c \) applied to the core [37]. The axial stress distribution in the core can be written as

\[ \sigma_y^c \geq \frac{3P^c L}{wb^2} \]  

(13)

Using the same method, Equation (13) can be rearranged in terms of core depth \( b \) and substituted into Equation (9) to find the performance index for a light strong core.

\[ m \geq (3P^c w)^{1/2} L^{3/2} \left( \frac{\rho^c}{\sqrt[3]{\sigma_y^c}} \right) \]  

(14)

In terms of maximising strength, the optimum lightweight core should therefore have a minimum value for \( \rho^c / \sqrt[3]{\sigma_y^c} \). The load carried by the core can be found by multiplying the externally applied load with the ratio of the core bending stiffness and the total bending stiffness.

\[ P^c = P \frac{(EI)^c}{(EI)^{eq}} \]  

(15)

5. Results and Discussion

5.1. Effect of Size Optimization

The effects of size optimisation for the individual lattice trusses on the compliance of the finite element models are presented in Table 3. For both diameter graded and spatially graded lattices the compliance reduces significantly following size optimization. It can be therefore be
established that this second level of optimisation is advantageous and a necessary step to achieve optimal structural compliance.

5.2. Sandwich Structure Performance

The experimental force-displacement characteristics of all three sandwich structures under three point bending are shown in Figure 8. The load values in this figure are without finite width corrections applied, and the lower surface deflections are calculated from the extensometer data. After finite width corrections, the experimental results show an increase in stiffness and strength of 30.1% and 119.4%, respectively, for the diameter graded structure compared to the benchmark sample, see Table 4. The improvement in performance of the spatially graded sandwich structure is even better with stiffness and strength increasing by 100.7% and 172.0%, respectively, when compared with the benchmark design.

The DIC results for vertical deflection are also provided in Figure 8 along with images showing the various failure modes observed in the sandwich structures. Initially, symmetric and uniform deflections with regard to the 3-point load application point were observed for all types of sandwich structures with minimal through-thickness core compression as evidenced by similar measured displacements at top and bottom centre points. The benchmark design was observed to undergo progressive core crushing initiating about the loading point. This type of failure mode is characteristic of sandwich structures and has been demonstrated several times in previous experimental studies [41, 42, 43]. The result of progressive core crushing can also be observed in Figure 8d whereby the initial linear stiffness transitions to highly nonlinear permanent deformation once failure is initiated. The diameter graded sample on the other hand has distinctively brittle response characteristics with relatively linear stiffness up to the point of catastrophic failure. Such a sudden failure indicates that the optimisation methods used in OptiStruct are successful in efficiently distributing stresses within the structure.

The force-displacement response of the spatially graded structure was also brittle with an almost perfectly linear response prior to failure. In comparison to the other two lattice structures, the spatially graded structure failed by local buckling of its longer lattice members in triangular cell configuration near the location of the applied load. The deformation of the spatially graded structure close to the point of buckling can be seen in Figure 8c with the DIC analysis capable of
detecting the onset of localised buckling as the lattice members start to rotate in an anti-clockwise manner.

5.3. Core Performance Indices

The core performance indices related to shear stiffness, Young’s modulus and yield strength are provided in Table 5. The non-dimensional core densities of the three designs are also provided, where \( \rho^c = 1 \) would represent the density of the parent material. After applying the relevant finite width correction factors, provided in Table 1, it was found that all three designs have core densities close to the target 25% volume fraction specified in the topology optimisation procedure. The similarity in core density and the similar total number of lattice cells makes the comparison between the three core designs as fair as possible. The small variations in core densities are primarily caused by geometry details such as finite model widths, the assumption of mid-plane symmetry, and other areas in the model such as the skin-core interface and lattice node details, where volumes of the finite elements overlap. All values presented in Table 5 are dimensionless since the analysis procedure presented in this work is independent of material properties, so long as the constituent material is isotropic with a Poisson’s ratio of 0.3.

The shear modulus performance indices \( G^c / \rho^c \) for all three core designs are significantly below unity. A value of 1 would indicate identical performance to the parent material. The poor shear performance of all three cores is not surprising as it is a result of the inherently poor shear characteristics of the BCC cell when compared with the equivalent bulk material. All three performance indices are below unity for the experimental benchmark results. The strength performance index of 0.12 is particularly low as a result of the early onset of localised core crushing. Similarly the experimental strength performance index for the spatially graded cell of 1.39 is also lower than expected as a result of the early onset of localised buckling. But otherwise the experimental and numerically predicted core performance indices show consistent trends with the experimental data.

5.4. Stress Distributions in Lattice Members

The maximum von Mises stress distributions within the three cores also show an interesting trend consistent with the experimental results. The stresses in Figure 9 are normalised with respect to the yield strength of the benchmark finite element model. Again, the benchmark and
diameter graded core models are shown to have near identical performance whereas the average von Mises stress in the spatially graded core is significantly reduced, along with the standard deviation in stresses. The reduction in the average stress and the standard deviation in the spatially graded core is due to orientating and sizing the lattice cells to reflect the positions of the theoretical isostatic lines. By using isostatic lines it should be theoretically possible to achieve a homogeneous von Mises stress distribution although in practice the discretisation of the isostatic lines into finite elements and other geometric details, such as model boundaries, will result in some variability.

6. Conclusions and Future Work

Sandwich structures containing cores comprising of BCC lattice cells were tested to failure under three point bending. Two optimised functionally-graded core designs were proposed based on lattice beam diameter tailoring and lattice cell spatial tailoring, respectively. Both stiffness and strength of the optimised cores are significantly increased compared to the uniform benchmark core. As expected, the greatest improvement in stiffness and strength was seen in the spatially graded lattice core due to the greater number of design variables available for optimisation. These additional degrees of design freedom are introduced using a novel isostatic line method developed in this research, which functionally grades the lattice cells in terms of size, aspect ratio and orientation to align the load-bearing truss members with the principal stresses within the core.

There were some discrepancies between the predicted and achieved performance, particularly in terms of failure load. The early onset of failure in the experimental benchmark structure from core crushing and the early failure of the spatially graded structure due to localised strut buckling cause the structures to fail prior to their predicted targets. Future work will involve the consideration of such failure modes during the modelling and optimisation phase. It is also planned to implement the isostatic line optimisation methodology for the case of three dimensional stress fields to validate additional engineering component case studies where using spatially graded lattices can lead to novel and more weight efficient designs.
Acknowledgements

The authors would like to acknowledge the support from the Agency for Science, Technology and Research (A*STAR) and Science and Engineering Research Council (SERC) of Singapore through the Additive Manufacturing Centre (AMC) Initiative – SIMTech-led R&D projects (SERC Grant No 142 6800088). The support of Dr H. Yang for printing of polymer parts for validation is acknowledged.

References


Table 1: Finite width correction factors for core material properties.

<table>
<thead>
<tr>
<th></th>
<th>Benchmark</th>
<th>Diameter graded</th>
<th>Spatially graded</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite Element ($w = l$)</td>
<td>0.635</td>
<td>0.587</td>
<td>0.472</td>
</tr>
<tr>
<td>Experiment ($w = 6l$)</td>
<td>0.998</td>
<td>0.978</td>
<td>0.932</td>
</tr>
</tbody>
</table>

Table 2: Influence of transverse shear flexibility.

<table>
<thead>
<tr>
<th>Geometry</th>
<th>$\frac{(EI)^{eq}}{\kappa L^2 (AG)^{eq}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark</td>
<td>5.63×10⁻²</td>
</tr>
<tr>
<td>Diameter graded</td>
<td>3.21×10⁻²</td>
</tr>
<tr>
<td>Spatially graded</td>
<td>1.51×10⁻¹</td>
</tr>
</tbody>
</table>

Table 3: Influence of size optimisation on FE model compliance.

<table>
<thead>
<tr>
<th></th>
<th>Normalised compliance $^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark</td>
<td>1.00</td>
</tr>
<tr>
<td>Diameter graded (before size optimisation)</td>
<td>0.92</td>
</tr>
<tr>
<td>Diameter graded (after size optimisation)</td>
<td>0.77</td>
</tr>
<tr>
<td>Spatially graded (before size optimisation)</td>
<td>0.68</td>
</tr>
<tr>
<td>Spatially graded (after size optimisation)</td>
<td>0.40</td>
</tr>
</tbody>
</table>

$^a$ No correction factors were applied
Table 4: Sandwich structure performance.

<table>
<thead>
<tr>
<th></th>
<th>Benchmark Test</th>
<th>FE</th>
<th>Diameter graded Test (% change)</th>
<th>FE  (% change)</th>
<th>Spatially graded Test (% change)</th>
<th>FE  (% change)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Max. load, $P_y$ [N]</td>
<td>631 $^a$</td>
<td>1203</td>
<td>1385 $^b$ (119.4%)</td>
<td>1474 (22.6%)</td>
<td>1717 $^c$ (172.0%)</td>
<td>1804 (50.0%)</td>
</tr>
<tr>
<td>Initial stiffness, $P/\delta$ [N/mm]</td>
<td>210</td>
<td>289</td>
<td>274 (30.1%)</td>
<td>334 (15.5%)</td>
<td>422 (100.7%)</td>
<td>528 (82.4%)</td>
</tr>
<tr>
<td>Deflection at max. load $\delta_y$ [mm]</td>
<td>6.56</td>
<td>4.16</td>
<td>7.37</td>
<td>4.41</td>
<td>5.00</td>
<td>3.42</td>
</tr>
</tbody>
</table>

$^a$ Failure by core crushing

$^b$ Brittle failure

$^c$ Buckling of lattice members near loading point

Table 5: Non-dimensional core performance indices.

<table>
<thead>
<tr>
<th></th>
<th>Benchmark Test ($\rho^c = 0.252$)</th>
<th>FE</th>
<th>Diameter graded Test ($\rho^c = 0.247$)</th>
<th>FE</th>
<th>Spatially graded Test ($\rho^c = 0.277$)</th>
<th>FE</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G^c/\rho^c$</td>
<td>0.16</td>
<td>0.22</td>
<td>0.49</td>
<td>0.36</td>
<td>0.18</td>
<td>0.21</td>
</tr>
<tr>
<td>$\sqrt[3]{EC^c}/\rho^c$</td>
<td>0.53</td>
<td>1.73</td>
<td>2.21</td>
<td>1.62</td>
<td>2.90</td>
<td>3.10</td>
</tr>
<tr>
<td>$\sqrt[3]{\sigma_y^c}/\rho^c$</td>
<td>0.12</td>
<td>1.03</td>
<td>1.12</td>
<td>1.09</td>
<td>1.39</td>
<td>2.11</td>
</tr>
</tbody>
</table>
Figure 1: BCC unit cell density range after size optimisation; (a) lower bound and (b) upper bound.

\[ \frac{\rho_c}{\rho} = 6.94 \times 10^{-2} \quad \frac{\rho_c}{\rho} = 6.94 \times 10^{-1} \]
Figure 2: Optimisation procedures to generate functionally graded lattice structures.
Figure 3: Construction of orthogonal isostatic lines.
Figure 4: Optimised core density distribution showing boundary conditions and beam dimensions.
Figure 5: Specimen designs: (a) uniform benchmark, (b) diameter graded and (c) spatially graded.
Figure 6: Spatially graded lattice topology: (a) optimal force line distribution based on topology optimised density distribution and (b) normalised axial stresses after size optimisation (beam diameters not to scale).
Figure 7: Three point bend test of spatially optimised core structure using digital image correlation.
Figure 8: Vertical deflection for (a) benchmark structure at end of test showing progressive core crushing \((u_z=9.5\text{mm})\); (b) diameter graded structure at ultimate load \((u_z=7.8\text{mm})\); (c) spatially graded structure at the onset of localised buckling \((u_z=4.8\text{mm})\) and; (d) experimental force-displacement results also indicating the positions of images a, b and c.
Figure 9: Normal distributions of maximum von Mises stress in the finite element models’ lattice members.